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SOME REMARKS ON SETS OF UNIFORM CONVERGENCE

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Let G be a compact infinite Abelian group whose dual group $\Gamma = \hat{G}$ is the union of an increasing sequence of finite subsets X_n. As in [7], a set E will be called a *set of uniform convergence*, or *UC-set*, if, whenever $f \in C_E(G)$ (the set of continuous functions f on G with $\hat{f}(\gamma) = 0$ for every $\gamma \notin E$), we have

$$\|S_n f - f\|_\infty \to 0, \quad \text{where } S_n f(x) = \sum_{\gamma \in X_n} \hat{f}(\gamma)(\gamma, x).$$

Figà-Talamanca [4] first exhibited UC-sets that are not Sidon sets [8], i.e., sets for which the Fourier series of some $f \in C_E(G)$ is not absolutely convergent; a detailed study of UC-sets is contained in [7].

The definition of UC-set depends on the choice of the sequence $\{X_n\}$ of subsets of Γ. We assume that this sequence has been chosen once for all. Whether the union of UC-sets is a UC-set and whether every set which is not Sidon contains a UC-set not Sidon are two interesting open questions. In this note, by means of a new construction of UC-sets, we give a sort of converse to the second problem. We prove that every Sidon set can be imbedded in a UC-set which is not Sidon; moreover, partial answers to the first question are given.

With the previous notation we have

PROPOSITION 1. *Let $A = \{\gamma_k\}_{k=1}^\infty$ be an infinite Sidon set and let $A_k = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. Then there exists a sequence $\{y_k\}_{k=1}^\infty$ in Γ such that*

$$E = \bigcup_{k=1}^{\infty} y_k A_k$$

is a UC-set but not a Sidon set. Moreover, for every Sidon set S, $E \cup S$ is a UC-set.

Proof. The well-known facts (see [8], Theorem 2.6.8) about approximate unity in $L^1(G)$ allow us to construct by induction a sequence $\{y_k\}_{k=1}^\infty$ of elements of Γ, a sequence $\{n_k\}_{k=1}^\infty$ of positive integers, and a sequence $\{P_k\}_{k=1}^\infty$ of trigonometric polynomials with the following properties:

(a) $\|P_k\|_{L^1(G)} \leqslant 2$,

(b) $\hat{P}_k(\gamma) = 1$ for every $\gamma \in X_{n_k}$,

(c) $\operatorname{supp}\hat{P}_k \subseteq X_{n_{k+1}}$,

(d) $\bigcup_{i=1}^{k} y_i A_i \subseteq X_{n_k}$,

(e) $y_{k+1}A_{k+1} \cap X_{n_k} = \emptyset$.

Let now $f \in C_{E \cup S}(G)$ and $n_k \leqslant n < n_{k+1}$. By (d) we have

$$S_n f(x) = f * P_{k+1}(x) - \sum_{\gamma \notin X_n} \hat{f}(\gamma)\hat{P}_{k+1}(\gamma)(\gamma, x). \tag{1}$$

Since, by (b), (c), (d), and (e),

$$\operatorname{supp}\hat{f}(\gamma)\left(\hat{P}_{k+1}(\gamma) - \hat{P}_{k-1}(\gamma)\right) \subseteq S \cup y_k A \cup y_{k+1} A,$$

by the Drury lemma [2] there exists a $\delta > 0$ independent of n and f and such that the following inequalities hold:

$$\begin{aligned} \Big\| \sum_{\gamma \notin X_n} \hat{f}(\gamma)\hat{P}_{k+1}(\gamma)(\gamma, x)\Big\|_\infty &\leqslant \sum_{\gamma \notin X_n} |\hat{f}(\gamma)\hat{P}_{k+1}(\gamma)| \\ &\leqslant \sum_{\gamma \in \Gamma} |\hat{f}(\gamma)|\,|\hat{P}_{k+1}(\gamma) - \hat{P}_{k-1}(\gamma)| \leqslant \delta\, \|f*(P_{k+1} - P_{k-1})\|_\infty. \end{aligned} \tag{2}$$

Hence by (a), (1), and (2) we have

$$\|S_n f\|_\infty \leqslant (2 + 4\delta)\|f\|_\infty.$$

From this inequality it follows easily that $S_n f$ converges to f uniformly. It is known that E cannot be a Sidon set (see [3], Theorem 3.5).

Remark. If $\Gamma = Z$ is the group of integers and $X_n = [-n, n]$, it is possible to replace in the construction of the set E the Sidon set A with a UC-set of positive integers. Indeed, choosing

$$X_{n_k} = [-y_{k-1} - \gamma_k, y_{k-1} + \gamma_{k-1}], \qquad y_k > 2(\gamma_{k-1} + y_{k-1})$$

and

$$P_k = V_{y_{k-1} + \gamma_{k-1}}$$

(where V_n is the de la Vallée-Poussin kernel [6]), we obtain, as in the previous proof,

$$\|S_n f\|_\infty \leqslant (2 + 4C)\|f\|_\infty \quad \text{for all } f \in C_E(T),$$

where C is the UC-constant of the set A (see [7]).

Since E is a positive UC-set, the construction can be iterated; moreover, since at the n-th iteration we obtain a UC-set containing the sum of n sets of arbitrarily large cardinality, by the lemma reported below we obtain the following result [9]: for every $1 \leqslant p < 2$, there exists a UC-set $E \subseteq Z$ which is not p-Sidon (that is, $\hat{f} \notin l^p(Z)$ for some $f \in C_E(T)$).

LEMMA (Johnson and Woodward [5]). *If $E \subseteq \Gamma$ is p-Sidon for some $1 \leqslant p < 2n/(n+1)$, then*

$$\operatorname{Sup}\{\min(|A_1|, |A_2|, \ldots, |A_n|): A_1+A_2+\ldots+A_n \subseteq E;\ A_1, A_2, \ldots, A_n \subseteq \Gamma\} < \infty.$$

In the context of the Remark, we prove the following proposition which extends a result from [1]. (For the definition of the Hadamard sets see [8].)

PROPOSITION 2. *The union of a UC-set F and a finite number of Hadamard sets is a UC-set.*

Proof. Let $H = \{n_k\}_{k=1}^{\infty}$ be a Hadamard set with $n_k > 0$ and $n_{k+1}/n_k \geqslant 4$. Let

$$V_n^* = \left(\frac{n+1}{[n/2]}+1\right)K_{n+[n/2]} - \frac{n+1}{[n/2]}K_n, \tag{3}$$

where $[a]$ is the integral part of a, and K_n is the Féjer kernel [6]. Then, if $f \in C_{F\cup H}(T)$, it is easy to see that

$$S_nf(x) = V_n^* * f(x) - \exp(in\cdot)K_{[n/2]}*f(x) + S_n\big(\exp(in\cdot)K_{[n/2]}*f\big)(x). \tag{4}$$

By the Hadamard condition, only one point in the set H lies in $\operatorname{supp}\exp(in\cdot)K_{[n/2]}*f$. Hence there exists a constant C (which depends only on F) such that

$$\|S_n\exp(in\cdot)K_{[n/2]}*f\|_\infty \leqslant (1+C)\|f\|_\infty. \tag{5}$$

Therefore, by (3), (4), and (5), we obtain $\|S_nf\|_\infty \leqslant (8+C)\|f\|_\infty$ and the proposition follows.

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