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CONCERNING IRREDUCIBLE CONTINUA WITH HOMEOMORPHIC LAYERS

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Let $\mathscr{K}$ denote the class of all compact metric continua K such that there exists an upper semicontinuous decomposition G of a compact metric irreducible continuum M with each element of G homeomorphic to K and the decomposition space M/G is an arc. Transue [6] has shown that no simple closed curve is in $\mathscr{K}$. It is shown in [4] that if n is a positive integer, then the n-cell is in $\mathscr{K}$, and the question (P 865) of whether the 2-sphere is in $\mathscr{K}$ is raised. Theorem 1 of this paper states that the n-sphere is not in $\mathscr{K}$ and Theorem 2 states the annulus is not in $\mathscr{K}$.

In this paper, I will denote the interval $[0, 1]$ and Q will denote the Hilbert cube. Also, $d(x, y)$ is the distance from x to y and $B(x, \varepsilon)$ is the set of all points y such that $d(x, y) < \varepsilon$ for some x in X. If G is a decomposition of a continuum M such that M/G is an arc, we will assume there is a function f mapping M into I such that

$$G = \{f^{-1}(t),\ t \in I\}.$$

Elements of G will be called *layers*. g is a *layer of continuity* if $f^{-1}\colon I \to 2^M$ is continuous at $f(g)$.

THEOREM 1. *No n-dimensional sphere is in $\mathscr{K}$.*

Proof. Assume the contrary. Let M denote a compact metric irreducible continuum such that there is a function f mapping M onto I such that $f^{-1}(x)$ for $x \in I$ is homeomorphic to an n-dimensional sphere S^n.

CLAIM. *For every non-empty open set U in I and for every positive integer k, there is an open and non-empty set $V(U, k)$ such that $\operatorname{cl} V(U, k)$ is a subset of U and, for every x in $V(U, k)$, there is a mapping $g_{xk}\colon f^{-1}(x) \to E^n$ such that the diameter of $g_{xk}^{-1}(y)$ is less than $1/k$ for every y in E^n.*

Proof of the Claim. By a theorem of Dyer [3], there is a point t in U such that $f^{-1}(t)$ is not a layer of continuity. It can be shown there are a point m in $f^{-1}(t)$ and a positive number ε such that for every $\delta > 0$ there is a number z in I such that

$$|z-t| < \delta \quad \text{and} \quad f^{-1}(z) \cap B(m, \varepsilon) = \emptyset.$$

Since $f^{-1}(t)$ is a sphere, there are a neighborhood W of $f^{-1}(t)$ in M and a retraction r of W to $f^{-1}(t)$. Without loss of generality, for each w in W,

$$d(w, r(w)) < \min(\varepsilon/2, 1/2k).$$

There is a positive number δ such that

$$f^{-1}([t-\delta, t+\delta]) \subset W \quad \text{and} \quad [t-\delta, t+\delta] \subset U.$$

Let z be such that $|z-t| < \delta$ and $f^{-1}(z) \cap B(m, \varepsilon) = \emptyset$. There is a neighborhood $V(U, k)$ of z such that $V(U, k) \subset (t-\delta, t+\delta)$ and, since r is continuous, for every x in $V(U, k)$ and every point p in $f^{-1}(x)$ there is a point q from $f^{-1}(z)$ such that $d(r(p), r(q)) < \varepsilon/2$. Clearly, m is not in the image of r restricted to $f^{-1}(x)$. Since $f^{-1}(t) - \{m\}$ is homeomorphic to E^n, we may assume $f^{-1}(t)$ is equal to E^n. Define g_{xk} as r restricted to $f^{-1}(x)$.

Let p_1 and p_2 denote two points of $f^{-1}(x)$ such that $r(p_1) = r(p_2)$. Since $d(w, r(w)) < 1/2k$ for every w in W and $f^{-1}(x)$ is contained in W, we have $d(p_1, r(p_1)) < 1/2k$. Therefore $d(p_1, p_2) < 1/k$. Hence $\operatorname{diam}(g_{xk}^{-1}(y)) < 1/k$ for every y in E^n. This completes the proof of the Claim.

We now proceed to complete the proof of Theorem 1. Let $U_1 = V(I, 1)$ and let $U_{k+1} = V(U_k, k+1)$. Obviously,

$$\bigcap_{k=1}^{\infty} U_k \neq \emptyset.$$

Suppose x is in $\bigcap_{k=1}^{\infty} U_k$. Let h denote a homeomorphism from $f^{-1}(x)$ onto S^n. There is a positive integer k such that, for each p_1 and p_2 from $f^{-1}(x)$ with $d(p_1, p_2) < 1/k$, the points $h(p_1)$ and $h(p_2)$ are not antipodal. But x is in $U_k = V(U_{k-1}, k)$. Consider the mapping $g_{xk} h^{-1}\colon S^n \to E^n$. According to the Borsuk–Ulam Theorem [2] there are antipodal points s and s^* such that $g_{xk} h^{-1}(s) = g_{xk} h^{-1}(s^*)$. It follows by the Claim that

$$d(h^{-1}(s), h^{-1}(s^*)) < 1/k.$$

This involves a contradiction. $d(p_1, p_2) < 1/k$, and hence $h(p_1)$ and $h(p_2)$ are not antipodal. This completes the proof.

THEOREM 2. *The annulus does not belong to* $\mathscr{K}$.

Proof. Let A denote an annulus and M denote a compact metric irreducible continuum such that there is a mapping f from M onto I such that, for each x in I, $f^{-1}(x)$ is homeomorphic to A. Suppose M is in Q.

DEFINITION. Let $I(\varepsilon)$ denote the set of all t from I such that the identity on $f^{-1}(t)$ is homotopic to a constant mapping in $B(f^{-1}(t), \varepsilon)$.

DEFINITION. Let $N(\varepsilon)$ denote the set of all t in I such that there is a $\delta > 0$ such that if $|z-t| < \delta$, then $f^{-1}(t) \subset B(f^{-1}(z), \varepsilon)$.

OBSERVATION 1. *For every $\varepsilon > 0$, the interior of $N(\varepsilon)$ is a dense set.*

Proof. Observe every element of continuity is contained in the interior of $N(\varepsilon)$. By Corollary 4, p. 72 of [5], the observation follows.

OBSERVATION 2. *$\bigcap_{\varepsilon>0} \mathrm{cl}\, I(\varepsilon)$ is nowhere dense in I.*

Proof. Assume the contrary. There is an open set $U \subset \mathrm{cl}\, I(\varepsilon)$ for $\varepsilon > 0$. We now construct a sequence $G_0, G_1, G_2, \ldots$ of open and non-empty sets so that, for each n,

$$\mathrm{cl}\, G_{n+1} \subset G_n \cap I\left(\frac{1}{n+1}\right).$$

Let $G_0 = U$. Suppose G_n is constructed. By Observation 1 we have

$$G_n \cap N\left(\frac{1}{2(n+1)}\right) \neq \emptyset.$$

Let

$$t \in G_n \cap N\left(\frac{1}{2(n+1)}\right) \cap I\left(\frac{1}{2(n+1)}\right) \neq \emptyset.$$

Combining Theorems 8.1 (p. 94) and 5.1 (p. 106) from [1], it can be seen there are a neighborhood V of $f^{-1}(t)$ in Q and a deformation $D\colon V \times I \to Q$ such that $D(v, 0) = v$ for all $v \in V$, $D(v, 1) \in f^{-1}(t)$ for all $v \in V$, and $D(x, s) = x$ for all x in $f^{-1}(t)$ and s in I.

There is a neighborhood V_0 of $f^{-1}(t)$ in V such that

$$\mathrm{diam}\, D(\{v\} \times I) < \frac{1}{2(n+1)}$$

for all v in V_0. Since t is in $I(1/2(n+1))$, there is a mapping

$$g\colon f^{-1}(t) \times I \to B\left(f^{-1}(t), \frac{1}{2(n+1)}\right)$$

such that $g(x, 0) = x$ for all x in $f^{-1}(t)$ and $g(f^{-1}(t) \times \{1\})$ is a one-point set. Since t is in $N(1/2(n+1))$, there is a neighborhood U_t in I such that, for each z in U_t,

$$f^{-1}(t) \subset B\left(f^{-1}(z), \frac{1}{2(n+1)}\right).$$

There is a neighborhood G_{n+1} of t in I such that

$$f^{-1}(G_{n+1}) \subset V_0 \quad \text{and} \quad \mathrm{cl}\, G_{n+1} \subset G_n \cap U_t.$$

In order to complete the construction, it suffices to show that

$$\mathrm{cl}\, G_{n+1} \subset I\left(\frac{1}{n+1}\right).$$

Let z be a point of $\mathrm{cl}\, G_{n+1}$. Define a homotopy $H\colon f^{-1}(z)\times I \to Q$ by

$$H(x, s) = \begin{cases} D(x, 2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ g\big(D(x, 1), 2s-1\big) & \text{for } 1/2 \leqslant s \leqslant 1. \end{cases}$$

H is a continuous homotopy between the identity on $f^{-1}(z)$ and a constant mapping. Now it suffices to prove that

$$H\big(f^{-1}(z)\times I\big) \subset B\left(f^{-1}(z), \frac{1}{n+1}\right).$$

We have $H(\{x\}\times I) = D(\{x\}\times I) \cup g\big(D(x, 1)\times I\big)$. By the definition of g,

$$g\big(D(x, 1)\times I\big) \subset B\left(f^{-1}(t), \frac{1}{2(n+1)}\right)$$

and, by the choice of U_t,

$$f^{-1}(t) \subset B\left(f^{-1}(z), \frac{1}{2(n+1)}\right).$$

Hence

$$g\big(D(x, 1)\times I\big) \subset B\left(f^{-1}(z), \frac{1}{n+1}\right).$$

Since x is in V_0, we have

$$D(\{x\}\times I) \subset B\left(x, \frac{1}{2(n+1)}\right) \subset B\left(f^{-1}(z), \frac{1}{n+1}\right).$$

This completes the construction.

Note that $\bigcap G_n \neq \emptyset$. Let t be a point in $\bigcap G_n$. By the construction, t belongs to $I(1/n)$ for every positive integer n. Therefore, the identity on $f^{-1}(t)$ is homotopic to a constant mapping in every neighborhood of $f^{-1}(t)$, which is impossible since $f^{-1}(t)$ is an annulus. This completes the proof of Observation 2.

Let ε denote a positive number such that

$$\mathrm{Int}\big[I - \mathrm{cl}\big(I(2\varepsilon)\big)\big] \neq \emptyset.$$

Without loss of generality, we may assume $I(2\varepsilon) = \emptyset$. Now, by Observation

1, the interior of $N(\varepsilon)$ is a dense set; without loss of generality we can also assume $N(\varepsilon) = I$.

OBSERVATION 3. *Let F be a subset of Q such that* $\operatorname{diam} F < \varepsilon$. *For every t in I, the identity on $f^{-1}(t) \cap F$ is homotopic to a constant mapping in $f^{-1}(t)$.*

Proof. Assume the contrary. Suppose there is a set F in Q such that $\operatorname{diam} F < \varepsilon$ and there is a $t \in I$ such that the identity on $f^{-1}(t) \cap F$ is not homotopic to a constant mapping in $f^{-1}(t)$.

There is a simple closed curve C contained in $f^{-1}(t) \cap B(F, \varepsilon/2)$ so that C is not contractible in $f^{-1}(t)$; that is to say, the identity on C is not homotopic to a constant mapping in $f^{-1}(t)$. Observe $\operatorname{diam} C < 2\varepsilon$. There is a homotopy $H: f^{-1}(t) \times I \to f^{-1}(t)$ such that $H(x, 0) = x$ and $H(x, 1)$ belongs to C for every x from $f^{-1}(t)$. But the identity on C is homotopic to a constant in the convex hull C_h of C. The composition of H and this homotopy give a homotopy between the identity on $f^{-1}(t)$ and a constant mapping in $f^{-1}(t) \cup C_h$. This contradicts the fact that $I(2\varepsilon) = \emptyset$, which completes the proof of Observation 3.

DEFINITION. Let $T(\delta)$ denote the set of all elements t in I such that there is a continuum $C_t(\delta) \subset f^{-1}(t)$ so that $\operatorname{diam} C_t(\delta) < \delta$ and $C_t(\delta)$ intersects both components of the boundary of $f^{-1}(t)$.

OBSERVATION 4. $\bigcap_{\delta > 0} \operatorname{cl} T(\delta)$ *is nowhere dense.*

Proof. Suppose $\bigcap_{\delta > 0} \operatorname{cl} T(\delta)$ contains an open and non-empty set U. We will now construct inductively a sequence $G_0, G_1, G_2, \ldots$ of open sets so that G_n is a non-empty open subset of U and

$$\operatorname{cl} G_{n+1} \subset G_n \cap T(1/n).$$

Let $G_0 = U$. Assume $G_0, G_1, \ldots, G_n$ are constructed. Since G_n is an open set in U, and $G_n \subset U \subset \operatorname{cl} T(1/2n)$, there is a $z \in G_n \cap T(1/2n)$. There is a continuum C in $f^{-1}(z)$ such that $\operatorname{diam} C < 1/2n$ and C intersects both components of the boundary of $f^{-1}(x)$. Again, combining Theorems 8.1 (p. 94) and 5.1 (p. 506) from [1], we infer that there are a neighborhood V of $f^{-1}(z)$ in the Hilbert cube and a homotopy $H: V \times I \to Q$ such that

(1) $H(x, s) = x$, $x \in f^{-1}(z)$ and $s \in I$,

(2) $H(v, 0) = v$ and $H(v, 1)$ is in $f^{-1}(z)$ for $v \in V$.

Recalling $N(\varepsilon) = I$, we may assume without loss of generality that

$$\operatorname{diam}\big(H(\{v\} \times I)\big) < \min(1/4n, \varepsilon). \tag{A$_1$}$$

Let G_{n+1} be an open neighborhood of z such that $\operatorname{cl} G_{n+1} \subset G_n$ and, for every t in $\operatorname{cl} G_{n+1}, f^{-1}(t) \subset V$ and

$$f^{-1}(z) \subset B\big(f^{-1}(t), \varepsilon\big). \tag{A$_2$}$$

Let F denote $\mathrm{cl}\{v \in V \mid$ there is a set $s \in I$ such that $H(v, s) \in C\}$. Observe $\operatorname{diam} F < 1/n$. Consider $f^{-1}(t) \cap F$. If for each t in $\mathrm{cl}\, G_{n+1}$ there is a component $f^{-1}(t) \cap F$ which intersects both components of the boundary of $f^{-1}(t)$, then the proof is complete.

Assume for some t in $\mathrm{cl}\, G_{n+1}$ no component of $f^{-1}(t) \cap F$ intersects both components of the boundary of $f^{-1}(t)$. Then $f^{-1}(t) - F$ contains a simple closed curve S, which is not contractible in $f^{-1}(t)$. Since $f^{-1}(t)$ is an annulus, there is a homotopy $H_1\colon f^{-1}(t) \times I \to f^{-1}(t)$ such that $H_1(x, 0) = x$ for all x in $f^{-1}(t)$ and $H_1(x, 1)$ belongs to S.

Let $H_2\colon (f^{-1}(z) - C) \times I \to f^{-1}(z)$ be a homotopy such that $H_2(x, 0) = x$ for all $x \in f^{-1}(z) - C$, and $H_2(x, 1)$ is constant for all x in $f^{-1}(z) - C$.

Let $H_3\colon f^{-1}(t) \times I \to Q$ be a homotopy defined by

$$H_3(x, s) = \begin{cases} H_1(x, 3s) & \text{for } 0 \leqslant s \leqslant 1/3, \\ H(H_1(x, 1), 3s-1) & \text{for } 1/3 \leqslant s \leqslant 2/3, \\ H_2[H(H_1(x, 1), 1), 3s-2] & \text{for } 2/3 \leqslant s \leqslant 1. \end{cases}$$

One can show that H_3 is a homotopy in $B(f^{-1}(t), \varepsilon)$. This is a contradiction because (A_1) and (A_2) imply that $I(\varepsilon) = \varnothing$. So the construction of $G_C, G_1, G_2, \ldots$ is complete.

Let t be in $\bigcap G_n$. The set $f^{-1}(t)$ is an annulus so that for each n there is a continuum with diameter less than $1/n$ meeting both components of the boundary of $f^{-1}(t)$. Hence the boundaries intersect. This involves a contradiction and the proof of Observation 4 is complete.

We now proceed to complete the proof of Theorem 2. Again without loss of generality, we may assume there is a positive number $\delta < \varepsilon$ such that $T(\delta) = \varnothing$. Let U be an open and non-empty set with diameter less than δ. For every t in I there is only one component C_t of $f^{-1}(t) - U$ such that the identity on C_t is not homotopic to a constant mapping in $f^{-1}(t)$.

Let

$$M_1 = \bigcup_{t \in I} C_t.$$

$f \mid M_1$ maps M_1 onto I and it is a monotone mapping. To prove M_1 is a continuum, it suffices to prove M_1 is compact. Suppose x is in $\mathrm{cl}\, M_1$. There is a point z such that x is in $f^{-1}(z)$. Suppose x does not belong to C_z. Let C be a component of $f^{-1}(z) - U$ which contains x. Since $C \neq C_z$, there is a homotopy $H\colon C \times I \to f^{-1}(z)$ so that $H(c, 0) = c$ and $H(c, 1)$ is a constant for all $c \in C$. There are a neighborhood V of C and a homotopy $H^*\colon V \times I \to Q$ so that $H^*(v, 0) = v$ and $H^*(v, 1)$ is constant for v in V and H^* restricted to $C \times I$ is H. Again, without loss of generality, the image of H^* is a subset of $B(f^{-1}(z), \varepsilon)$.

Since x belongs to $\mathrm{cl}\, M_1$ and components of $M - U$ form an upper semicontinuous decomposition of $M - U$, there is a point t^* in I such that

C_{t^*} is a subset of V. Since $I = N(\varepsilon)$, without loss of generality $f^{-1}(z) \subset B(f^{-1}(t^*), \varepsilon)$. Since only one component of $f^{-1}(t^*) - \mathrm{cl}\, U$ is not contractible in $f^{-1}(t^*)$ and this component is contained in C_{t^*}, there is a simple closed curve J in C_{t^*} which is not contractible in $f^{-1}(t^*)$.

Let H_1 be a homotopic mapping of $f^{-1}(t^*) \times I \to f^{-1}(t^*)$ so that $H_1(y, 0) = y$ and $H_1(y, 1)$ is in J for all y in $f^{-1}(t^*)$.

Let $H_2\colon f^{-1}(t^*) \times I \to Q$ be defined by

$$H_2(y, x) = \begin{cases} H_1(y, 2s) & \text{for } 0 \leqslant s \leqslant 1/2, \\ H^*(H_1(y, 1), 2s-1) & \text{for } 1/2 \leqslant s \leqslant 1. \end{cases}$$

Observe that $H_2(y, 0) = y$ and $H_2(y, 1)$ is a constant for every y in $f^{-1}(t^*)$. The image of H_2 is a subset of the union of the image of H_1 and the image of H^*, which is a subset of $f^{-1}(t^*) \cup B(f^{-1}(z), \varepsilon)$. Since

$$f^{-1}(z) \subset B(f^{-1}(t^*), \varepsilon) \subset B(f^{-1}(t^*), 2\varepsilon),$$

the image of H_2 is a subset of $B(f^{-1}(t^*), 2\varepsilon)$, which is impossible, due to the assumption after Observation 2 that $I(2\varepsilon) = \emptyset$.

Hence $\bigcup_{t=I} C_t$ is compact and is a proper subcontinuum of M intersecting $f^{-1}(0)$ and $f^{-1}(1)$. This involves a contradiction and the proof of Theorem 2 is complete.

QUESTIONS. Is there a closed manifold in $\mathscr{K}$? Does $\mathscr{K}$ contain any closed manifold or the Cartesian product of a closed manifold with an arc? **(P 1325)**

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