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REMARK ON SUMS OF COMPLEMENTED SUBSPACES

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In [1], Edelstein and Wojtaszczyk have proved that every complemented subspace of a finite direct sum of totally incomparable Banach spaces is isomorphic to a finite direct sum of complemented subspaces of the summands. One of the essential facts in their proof was the following special case of a result of Kato ([2], Theorems 7 and 8):

If Φ is a Fredholm operator of index k, and Ψ is a strictly singular operator (cf. [1] for definitions), then $\Phi+\Psi$ is likewise a Fredholm operator of index k.

This fact can also be used to give a simple proof of a result related to that in [1].

PROPOSITION. *If P and Q are projections on a Banach space X, and PQ is strictly singular, then $PX+QX$ is complemented in X.*

Proof. We can assume that $PX\cap QX = \{0\}$. Write $T = I-PQ$. Then T is a Fredholm operator of index 0. Hence $\ker T$ is a finite-dimensional subspace of PX. Since $TPX \subset PX$, PX/TPX is also finite-dimensional. Let E be a complement of TPX in PX. If $x \in X$ and $Tx \in E$, then $x = Tx+PQx \in PX$, but $TPX\cap E = \{0\}$. Thus $E\cap TX = \{0\}$.

Since

$$\infty > \dim(X/TX) = \dim\ker T = \dim(PX/TPX) = \dim E,$$

E is a complement of TX in X. Let φ be an isomorphism of $\ker T$ onto E. Since $\ker T$ is finite-dimensional and $\ker T\cap QX = \{0\}$, there exists a projection S of X onto $\ker T$ annihilating QX.

Put

$$U = \varphi S+T(I-S) = \varphi S+T.$$

Since T maps any complement of $\ker T$ isomorphically onto TX, U is an isomorphism of X onto itself.

Now

$$UQ = \varphi SQ+TQ = TQ = (I-P)Q,$$

so $UQX \subset (I-P)X$. On the other hand, from $UP = \varphi SP+TP$ it follows that $UPX = PX$. Thus $UPX+UQX$ is complemented in X.

COROLLARY. *A finite sum of totally incomparable complemented subspaces is complemented. Also, if A and B are complemented subspaces of X and every bounded linear operator from A into B is compact, then $A+B$ is complemented in X.*

REFERENCES

[1] I. S. Edelstein and P. Wojtaszczyk, *On projections and unconditional bases in direct sums of Banach spaces*, Studia Mathematica 56 (1976), p. 263-276.

[2] T. Kato, *Perturbation theory for nullity, deficiency, and other quantities of linear operators*, Journal d'Analyse Mathématique 6 (1958), p. 261-322.

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