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DECOMPOSABILITY OF POINT MEASURES IN GENERALIZED CONVOLUTION ALGEBRAS

BY

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Throughout this paper, $\mathfrak{P}$ denotes the set of all Borel probability measures on the positive half-line R_+ endowed with the topology of weak convergence. The measure concentrated at a single point c will be denoted by δ_c. Further, by T_c $(c > 0)$ we denote the map defined by the formula

$$(T_c \mu)(E) = \mu(c^{-1} E)$$

for $\mu \in \mathfrak{P}$ and Borel subsets E of R_+.

Generalized convolutions were introduced in [2]. Let us recall some definitions. A continuous in each variable separately commutative and associative $\mathfrak{P}$-valued binary operation $\circ$ on P is called a *generalized convolution* if it is distributive with respect to convex combinations and maps T_c $(c > 0)$ with δ_0 as the unit element. Moreover, it is assumed that for a certain sequence c_n of norming constants and a measure γ different from δ_0 the relation

$$T_{c_n} \delta_1^{\circ n} \to \gamma \tag{1}$$

holds. Here $\delta_1^{\circ n}$ is the n-th power of δ_1 under $\circ$. Recently, Urbanik proved in [3] that each generalized convolution is continuous in both variables.

A generalized convolution is said to be *quasi-regular* if the norming sequence c_n in (1) tends to 0. This concept was introduced in [1].

A measure λ from $\mathfrak{P}$ is said to be *decomposable* under a generalized convolution $\circ$ if $\lambda = \mu \circ \nu$ for some μ and ν from $\mathfrak{P}$ different from δ_0. Let $0 < \alpha \leqslant \infty$. For any pair $\mu, \nu \in \mathfrak{P}$ we denote by $\mu \circ_\alpha \nu$ the probability distribution of $(X^\alpha + Y^\alpha)^{1/\alpha}$ if $0 < \alpha < \infty$ and $\max(X, Y)$ if $\alpha = \infty$, where the random variables X and Y are independent and have the probability distributions μ and ν, respectively. It is clear that $\circ_\alpha$ are generalized convolutions and

$$\delta_1 = \delta_a \circ_\alpha \delta_b \quad \text{if } a^\alpha + b^\alpha = 1 \ (0 < \alpha < \infty),$$
$$\delta_1 = \delta_a \circ_\infty \delta_1 \quad \text{if } a \leqslant 1.$$

The generalized convolution $\circ_\alpha$ is called the *α-convolution*. The last equations show that the measure δ_1 is decomposable under α-convolutions. K. Urbanik asked (see [1], P 827) whether the converse implication is true. We shall answer this question in the affirmative. Namely, we prove the following

THEOREM. *If the measure δ_1 is decomposable under a generalized convolution $\circ$, then $\circ$ is an α-convolution.*

Before proving the Theorem we prove some lemmas.

By $N(\mu)$ we denote the support of the measure μ. We start with the following useful remark:

LEMMA 1. *If $\delta_1 = \mu \circ \nu$, then*

$$\delta_1 = \delta_a \circ \delta_b = \delta_a \circ \nu \quad \textit{for all } (a, b) \in N(\mu) \times N(\nu).$$

Proof. The measure $\mu \circ \nu$ has an integral representation

$$\mu \circ \nu = \int_0^\infty \int_0^\infty \delta_a \circ \delta_b \, \mu(da) \, \nu(db),$$

where the integral is taken in the weak sense ([3], formula (2.13)). By Proposition 2.3 in [3],

$$N(\delta_a \circ \delta_b) \subset N(\mu \circ \nu), \qquad N(\delta_a \circ \nu) \subset N(\mu \circ \nu)$$

for all $(a, b) \in N(\mu) \times N(\nu)$. If $\delta_1 = \mu \circ \nu$, then

$$N(\delta_a \circ \delta_b) \subset \{1\} \quad \text{and} \quad N(\delta_a \circ \nu) \subset \{1\}$$

for all $(a, b) \in N(\mu) \times N(\nu)$. In other words,

$$\delta_1 = \delta_a \circ \delta_b = \delta_a \circ \nu$$

for all pairs (a, b) in question, which completes the proof.

LEMMA 2. *If $\delta_1 = \delta_a \circ \mu$ for a certain measure μ, then $a \leqslant 1$.*

Proof. By the induction we get easily the formula

$$\delta_1 = \delta_{a^n} \circ \mu_n \quad (n = 1, 2, \ldots), \tag{2}$$

where

$$\mu_n = \mathop{\bigcirc}_{j=0}^{n-1} T_{a^j} \mu \quad (n = 1, 2, \ldots).$$

Suppose that $a > 1$. Then $\delta_{a^{-n}} \to \delta_0$. On the other hand, by (2),

$$\delta_{a^{-n}} = \delta_1 \circ T_{a^{-n}} \mu_n \quad (n = 1, 2, \ldots),$$

which by Corollary 2.4 in [3] yields the contradiction $\delta_1 \to \delta_0$. Thus $a \leqslant 1$, which completes the proof.

LEMMA 3. *For a quasi-regular generalized convolution the equation $\delta_1 = \delta_1 \circ \mu$ yields $\mu = \delta_0$.*

Proof. Suppose the contrary $\mu \neq \delta_0$. Then, by Lemma 1, $\delta_1 = \delta_1 \circ \delta_a$ for a positive number a from $N(\mu)$. By a simple induction we have the formula

$$\delta_1 = \delta_1 \circ \delta_a^{\circ n} \quad (n = 1, 2, \ldots).$$

Taking the norming sequence c_n in (1) we have also

$$\delta_{c_n} = \delta_{c_n} \circ T_a T_{c_n} \delta_1^{\circ n} \quad (n = 1, 2, \ldots).$$

Since $\delta_{c_n} \to \delta_0$, we get the equation $\delta_0 = T_a \gamma$ when $n \to \infty$. Consequently, $\gamma = \delta_0$, which gives a contradiction. The lemma is thus proved.

From Lemmas 2 and 3 we obtain the following

COROLLARY. *For a quasi-regular generalized convolution the equation $\delta_1 = \delta_a \circ \delta_b$ for some positive numbers a and b yields $a < 1$ and $b < 1$.*

LEMMA 4. *Suppose that $\varrho(n, m)$ $(n, m = 0, 1, 2, \ldots)$ is an array of probability measures from $\mathfrak{P}$ with the properties*

$$\varrho(n, m) = \varrho(m, n), \quad \varrho(n, m) = \varrho(n+1, m) \circ \varrho(n, m+1) \tag{3}$$

$(n, m = 0, 1, 2, \ldots)$. *Then*

$$\varrho(0, 1) = \varrho(1, k-1)^{\circ k} \circ \nu_k \quad (k = 3, 4, \ldots), \tag{4}$$

where $\nu_3 = \varrho(0, 3)$ and

$$\nu_k = \varrho(0, k) \circ \mathop{\bigcirc}_{j=3}^{k-1} \varrho(2, j-1)^{\circ j} \quad (k > 3).$$

Proof. We prove our statement by induction. Using (3) we have

$$\begin{aligned}\varrho(0, 1) &= \varrho(1, 1) \circ \varrho(0, 2) = \big(\varrho(2, 1) \circ \varrho(1, 2)\big) \circ \big(\varrho(1, 2) \circ \varrho(0, 3)\big) \\ &= \varrho(1, 2)^{\circ 3} \circ \varrho(0, 3),\end{aligned}$$

which shows that (4) is true for $k = 3$. Suppose now that (4) is true for some $k \geqslant 3$. Using (3) we have

$$\varrho(1, k-1) = \varrho(2, k-1) \circ \varrho(1, k), \quad \varrho(0, k) = \varrho(1, k) \circ \varrho(0, k+1).$$

Thus

$$\varrho(1, k-1)^{\circ k} \circ \varrho(0, k) = \varrho(1, k)^{\circ (k+1)} \circ \varrho(0, k+1) \circ \varrho(2, k-1)^{\circ k},$$

which together with (4) yields $\delta_1 = \varrho(1, k)^{\circ (k+1)} \nu_{k+1}$. This completes the proof.

LEMMA 5. *Suppose that $\mu \in \mathfrak{P}$, $k \geqslant 1$, $\delta_1 = \mu^{\circ k} \circ \delta_a$, and $a < 1$. Then there exists a measure $\lambda \in \mathfrak{P}$ such that $\delta_1 = \lambda^{\circ k}$.*

Proof. By a simple induction we get the formula

$$\delta_1 = \mu_n^{\circ k} \circ \delta_{a^n} \quad (n = 1, 2, \ldots),$$

where

$$\mu_n = \mathop{\bigcirc}_{j=0}^{n-1} T_{a^j}\mu \quad (n = 1, 2, \ldots).$$

By Corollary 2.3 in [3] the sequence μ_n is conditionally compact in $\mathfrak{P}$. Let λ be its limit point. Since $\delta_{a^n} \to \delta_0$, we have then $\delta_1 = \lambda^{\circ k}$, which completes the proof.

Proof of the Theorem. If the operation $\circ$ is not quasi-regular, then, by Theorem 4.1 in [3], $\circ = \circ_\infty$. Consider the case of quasi-regular generalized convolutions. Let $\delta_1 = \mu \circ \nu$ with μ and ν different from δ_0. Then, by Lemma 1,

$$\delta_1 = \delta_a \circ \delta_b \tag{5}$$

for some positive numbers a and b. Moreover, by the Corollary to Lemma 3, $a < 1$ and $b < 1$. From (5), by the distributivity of the operation $\circ$ with respect to all maps T_c $(c > 0)$, we get the equations

$$\delta_{a^n b^m} = \delta_{a^{n+1} b^m} \circ \delta_{a^n b^{m+1}} \quad (n, m = 0, 1, 2, \ldots). \tag{6}$$

Put

$$\varrho(n, m) = \delta_{a^n b^m} \circ \delta_{a^m b^n} \quad (n, m = 0, 1, 2, \ldots).$$

It is easy to show, by virtue of (6), that the measures $\varrho(n, m)$ fulfil the conditions of Lemma 4 and $\varrho(0, 1) = \delta_1$. Consequently,

$$\delta_1 = \varrho(1, k-1)^{\circ k} \circ \nu_k \quad (k = 3, 4, \ldots),$$

where $\nu_3 = \varrho(0, 3)$ and

$$\nu_k = \varrho(0, k) \circ \mathop{\bigcirc}_{j=3}^{k-1} \varrho(2, j-1)^{\circ j} \quad (k > 3).$$

Moreover, by Lemma 2.3 in [3], the inequalities $a > 0$ and $b > 0$ yield $\varrho(n, m) \neq \delta_0$ $(n, m = 0, 1, 2, \ldots)$. Consequently, $\nu_k \neq \delta_0$ $(k = 3, 4, \ldots)$. Applying Lemma 1, we conclude that

$$\delta_1 = \varrho(1, k-1)^{\circ k} \circ \delta_{a_k}$$

for a certain positive number a_k belonging to $N(\nu_k)$. Furthermore, by Lemmas 2 and 3, $a_k < 1$ $(k = 3, 4, \ldots)$, which by Lemma 5 gives the existence of the measure λ_k satisfying the equation $\delta_1 = \lambda_k^{\circ k}$ $(k = 3, 4, \ldots)$. Thus, in other words, the measure δ_1 is infinitely divisible in the sense of the generalized convolution $\circ$. Now, our assertion is a direct consequence of the Theorem in [1], p. 142, which completes the proof.

REFERENCES

[1] J. Kucharczak, *A characterization of α-convolutions*, Colloq. Math. 27 (1973), pp. 141–147.
[2] K. Urbanik, *Generalized convolutions*, Studia Math. 23 (1964), pp. 217–245.
[3] — *Quasi-regular generalized convolutions*, Colloq. Math., this fasc., pp. 147–162.

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