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METRIC PROPERTIES OF SOME PLANAR SETS

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The graph of a continuous function defined on a linear interval is a planar set of Lebesgue measure 0. In this note we prove a theorem about differentiable mappings on certain subsets of the plane, and derive from it the following statement:

THEOREM 1. *Let h be a continuous increasing function on* $(0, +\infty)$ *and let*

$$\lim_{t\to 0+} h(t)t^{-1} = \infty.$$

Then there is a continuous functions f on $[0, 1]$, *and a Baire probability measure μ concentrated on the graph of f, so that $\mu(E) \leqslant h^2(\operatorname{diam} E)$ for open sets E.*

Theorem 1 answers a question proposed by Ralph Alexander. The proof stated here illustrates a method in abstract analysis; quite possibly a more "elementary" argument exists, but the details would be difficult to execute. The method is that of [4]; see also [1], [2] and [5].

1. It is convenient to write $x\cdot y$ for the scalar product in R^2, and $e(u) = e^{2\pi i u}$ for real numbers u.

A compact set F in R^2 is called a *Dirichlet set* [2] if there exist two sequences (X_n), (Y_n) in R^2 so that

(i) $\lim \|X_n\| = +\infty, \lim \|Y_n\| = +\infty$,

(ii) $\|X_n\| \leqslant \|Y_n\| < C\|X_n\|$ for some constant C,

(iii) $|X_n\cdot Y_n| \leqslant (1-\delta)\|X_n\|\cdot\|Y_n\|$ for some $\delta > 0$,

(iv) $\lim e(X_n\cdot x) = 1, \lim e(Y_n\cdot x) = 1$ uniformly with respect to x in F.

2. The theorem on transformations of planar Dirichlet sets—from which Theorem 1 will be easily derived—relates to a Banach space of differentiable functions. Let C^1 be the space of real continuously differentiable functions on R^2 with finite norm:

$$\|\varphi\| = \sup|\varphi| + \sup\|\operatorname{grad}\varphi\|.$$

THEOREM 2. *Let φ_0 be an element of C^1 and $I(\varphi_0)$ be the subset of C^1 defined by*

$$\psi \in C^1, \quad \operatorname{grad}(\psi - \varphi_0) = 0 \text{ on } F.$$

Then $I(\varphi_0)$ contains an element ψ_0 that is 1-1 on F; indeed, the elements ψ_0 are C^1-dense in $I(\varphi_0)$.

Proof. Let (ε_n) be a sequence decreasing to 0 so that

$$|e(X_n \cdot x) - 1| < \varepsilon_n, \quad |e(Y_n \cdot x) - 1| < \varepsilon_n \quad \text{for all } x \text{ in } F.$$

Then F is contained in the union of cells of the following type:

$$(Q_n(u, v)) \quad x \in R^2, \quad |X_n \cdot x - u| < \varepsilon_n, \quad |Y_n \cdot x - v| < \varepsilon_n.$$

Here (u, v) is a pair of integers.

For large n the cells Q_n have diameters $\ll \|X_n\|^{-1}\varepsilon_n$, by virtue of (ii) and (iii), and have mutual distances $\gg \|X_n\|^{-1}$.

For each pair (F_1, F_2) of disjoint closed subsets of F, let

$$N(F_1, F_2) = \{\psi \in I(\varphi_0) : \psi(F_1) \cap \psi(F_2) \neq \emptyset\}.$$

Then $N(F_1, F_2)$ is closed; as soon as we have proved that each set $N(F_1, F_2)$ is meager in $I(\varphi_0)$, Theorem 2 then follows by an argument of countability.

Let n be so large that no cell Q_n meets both F_1 and F_2. For each integer pair we define a number $r(u, v)$ as follows: when $Q_n(u, v)$ meets F_1, $r(u, v)$ is the *smallest* real number r such that $\varphi_0(Q_n(u, v)) + r$ contains a number of the form $2w\|X_n\|^{-1}\varepsilon_n^{1/2}$, $w \in Z$. When $Q_n(u, v)$ meets F_2, we consider numbers of the form $(2w+1)\ \|X_n\|^{-1}\varepsilon_n^{1/2}$; for all other cells, $r(u, v) = 0$. In every case $|r| \leqslant \|X_n\|^{-1}\varepsilon_n^{1/2}$.

Observe that the transforms $\varphi_0(Q_n(u, v))$ have diameters $\ll \|\varphi_0\|\ \|X_n\|^{-1}\varepsilon_n$, so that for large n the sets G_i defined for $i = 1, 2$ by

$$G_i = \bigcup \varphi_0(Q_n(u, v)) + r(u, v) : Q_n(u, v) \cap F_i \neq \emptyset$$

are disjoint, and have distance $\gg \|X_n\|^{-1}\varepsilon_n^{1/2}$.

Let d_n be the common diameter of all the cells Q_n and let $P_n(u, v)$ be the ball of radius $4d_n$ and the same center as $Q_n(u, v)$. Define $r_1 = r(u, v)$ on $P_n(u, v)$; then for all x and x' in the domain of r_1,

$$|r_1(x) - r_1(x')| \ll \varepsilon_n^{1/2}\ \|x - x'\|.$$

By a well-known technique, r_1 can be extended to a function r_2 on R^2 such that

$$\sup|r_2| = \sup|r_1| \leqslant \|X_n\|^{-1}\varepsilon_n^{1/2},$$

$$|r_2(x) - r_2(x')| \ll \varepsilon_n^{1/2}\|x - x'\|.$$

Define finally

$$r_3(x) = \pi^{-1} d_n^{-2} \iint_{|y|<d_n} r_3(x-y)\,dy.$$

Then $r_3 \in C^1$ and in this space has norm $\leqslant \varepsilon_n^{1/2}$; $r_3 = r$ on $\bigcup P_n$ so that grad $r_3 = 0$ on F; and $\varphi_0 + r_3 \notin N(F_1, F_2)$. This proves our theorem.

Choosing an element φ_0 of C^1 so that $\varphi_0(x_1, x_2) = x_1$ on an open square enclosing F, we obtain

COROLLARY. *There exists a continuously differentiable function ψ on R^2, with* $\mathrm{grad}(\psi - \varphi_0) = 0$ *on F and* $\|\mathrm{grad}(\psi - \varphi_0)\| < \frac{1}{2}$ *in R^2, that is* 1-1 *on F.*

Observe then that the mapping of R^2 onto R^2 defined by

$$H(x_1, x_2) \equiv \big(\psi(x_1, x_2), x_2\big)$$

has Jacobian matrix everywhere non-singular. Thus the mapping

$$\psi(x_1, x_2) \to x_2, \qquad (x_1, x_2) \in F,$$

on a compact subset of R, has the graph $H(F)$.

3. To prove Theorem 1, let us suppose that there is a measure μ of the required kind concentrated on a Dirichlet set F, and that H is the mapping of the last paragraph. Then the measure $\mu^+ = \mu \circ H^{-1}$, defined by

$$\mu^+(E) = \mu\big(H^{-1}(E)\big) \quad \text{for open sets } E,$$

has the property imposed in μ, while $\mu^+\big(H(F)\big) = \mu^+(R^2) = 1$.

To complete the argument we must construct a suitable measure μ on a Dirichlet set F. This is rather easy by standard techniques in descriptive set theory [3]. We have only to choose a sequence ε_k tending to 0 sufficiently slowly, and a sequence n_k of integers increasing sufficiently rapidly, and define F by the system of inequalities

$$0 \leqslant x_1,\ x_2 \leqslant 1, \qquad |e(n_k x_1) - 1| < \varepsilon_k, \qquad |e(n_k x_2) - 1| < \varepsilon_k.$$

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