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A PROBLEM OF INVARIANCE FOR LEBESGUE MEASURE

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Let $\varrho(\cdot, \cdot)$ be a metric on $\boldsymbol{R}^n$ which is invariant under translations, i.e.,

$$\varrho(x+z, y+z) = \varrho(x, y) \quad \text{for all } x, y, z \in \boldsymbol{R}^n,$$

and induces the usual topology of $\boldsymbol{R}^n$, i.e.,

$$\lim_{\|x\|\to 0} \varrho(0, x) = 0 \quad \text{and} \quad \lim_{\varrho(0,x)\to 0} \|x\| = 0,$$

where $\|\cdot\|$ is the usual Euclidean norm in $\boldsymbol{R}^n$.

For any sets $A, B \subseteq \boldsymbol{R}^n$ we say that A is *ϱ-isometric* to B if there exists a bijection $f\colon A \to B$ such that $\varrho(f(x), f(y)) = \varrho(x, y)$ for all $x, y \in A$.

$\lambda(\cdot)$ denotes the n-dimensional Lebesgue measure in $\boldsymbol{R}^n$.

It was proved in [1] that

(*) For every two open sets $A, B \subseteq \boldsymbol{R}^n$, if A is ϱ-isometric to B, then $\lambda(A) = \lambda(B)$.

PROBLEM. Is (*) true for all Borel sets $A, B \subseteq \boldsymbol{R}^n$? **(P 1085)**

We do not know the answer even for $n = 1$. We will prove that the answer is positive under additional conditions on ϱ.

THEOREM. *If there exist constants $a, \beta > 0$ such that for every open ϱ-ball B of diameter less than or equal to β there exists a parallelotope $P \subseteq B$ such that $\lambda(P) \geqslant a\lambda(B)$, then ϱ-isometric Borel sets have the same Lebesgue measure.*

Proof. Let C be the unit cube in $\boldsymbol{R}^n$. Let $\mathscr{B}$ be the family of all open ϱ-balls in $\boldsymbol{R}^n$. For any $t > 0$, let $E(t)$ be the least number of balls in $\mathscr{B}$ of diameter t necessary to cover C, and put $h(t) = 1/E(t)$. Now, with coverings from $\mathscr{B}$, we define the Hausdorff h-measure μ_h on $\boldsymbol{R}^n$ (see [3]).

Clearly, ϱ-isometric Borel sets have the same μ_h-measure. To prove our theorem it is enough to show that $0 < \mu_h(C) < \infty$. In fact, by the uniqueness of Haar measure, this implies $\mu_h = c\lambda$ for some constant $c > 0$, and hence the ϱ-invariance of λ.

First, for any $t > 0$ there exist balls $B_1, \ldots, B_{E(t)} \in \mathscr{B}$ of ϱ-diameter t with $C \subseteq B_1 \cup \ldots \cup B_{E(t)}$. Hence

$$\mu_h(C) \leqslant \sum_{1}^{E(t)} h(t) = 1.$$

Next, let $B_1, \ldots, B_n \in \mathscr{B}$ cover C, and let the ϱ-diameter of B_i equal t_i. If P is a parallelotope of sufficiently small diameter, then C can be covered with less than $2/\lambda(P)$ translates of P. Hence, if t_i is sufficiently small, then

$$E(t_i) < 2/\lambda(P_i) \leqslant 2/\big(a\lambda(B_i)\big),$$

where P_i is the parallelotope in B_i given by the assumption of the theorem. Thus

$$h(t_i) > a\lambda(B_i)/2.$$

Since $\sum_{1}^{n} \lambda(B_i) \geqslant 1$, we have

$$\sum_{1}^{n} h(t_i) \geqslant a/2.$$

Therefore $\mu_h(C) \geqslant a/2$, which completes our proof.

COROLLARY 1. *If there exist constants $a, \beta > 0$ such that for every ϱ-ball B of diameter less than or equal to a there exist Euclidean balls B_0 and B_1 of Euclidean diameters r_0 and r_1, respectively, with*

$$B_0 \subseteq B \subseteq B_1 \quad \textit{and} \quad r_0/r_1 \geqslant \beta,$$

then ϱ-isometric Borel sets have the same Lebesgue measure.

For the proof it is enough to check that the assumptions of the Theorem are satisfied.

The function $\varrho(0, x)$ is continuous, but it could be very irregular at 0. We know only one positive property:

PROPOSITION. *There exist $a, \beta > 0$ such that $\varrho(0, x) \geqslant \beta \|x\|$ whenever $\|x\| < a$.*

Proof. Let $\varepsilon, \eta > 0$ be arbitrary and choose $\delta > 0$ such that $\|x\| < \delta$ implies $\varrho(0, x) < \varepsilon$. Assuming the proposition false, we infer that there exists an x with $0 < \|x\| < \delta$ and $\varrho(0, x) < \eta \|x\|$. Then, by the triangle inequality,

$$\varrho(0, [1/\|x\|]x) \leqslant [1/\|x\|]\varrho(0, x) \leqslant [1/\|x\|]\eta\|x\| \leqslant \eta,$$

where $[\cdot]$ is the greatest integer function. Hence

$$\varrho(0, x/\|x\|) \leqslant \varrho(0, [1/\|x\|]x) + \varepsilon \leqslant \eta + \varepsilon.$$

It follows that there exist $y \in S^{n-1}$ (unit sphere in $\boldsymbol{R}^n$) with $\varrho(0, y)$ as small as we wish. From compactness of S^{n-1} and continuity of ϱ it follows that $\varrho(0, y) = 0$ for some $y \in S^{n-1}$. But this is a contradiction.

COROLLARY 2. *If there exist constants γ, $\delta > 0$ such that*

$$\varrho(0, x) \leqslant \gamma \|x\| \quad \text{for } \|x\| < \delta \tag{1}$$

or else there exists an $a > 0$ such that

$$\varrho(0, x) < \varrho(0, y) \quad \text{for } \|x\| < \|y\| < a, \tag{2}$$

then ϱ-isometric Borel sets have the same Lebesgue measure.

Proof. Assume (1). Then, by the Proposition, there exist $\sigma, \tau > 0$ such that

$$\sigma\|x\| \leqslant \varrho(0, x) \leqslant \gamma\|x\| \quad \text{for } \|x\| < \tau.$$

Thus Corollary 1 applies with $a = \tau$ and $\beta = \sigma/\gamma$.

Assume (2). Then open ϱ-balls of sufficiently small diameters are Euclidean, and Corollary 1 applies again. Thus the proof is completed.

Corollary 2 applies to such familiar metrizations of $\boldsymbol{R}^n$ as metrizations by homogeneous norms or metrizations of the form

$$\frac{\|x-y\|}{1+\|x-y\|} \quad \text{or} \quad \sqrt{\|x-y\|}.$$

However, it is easy to produce metrizations to which Corollary 1 is applicable but Corollary 2 is not.

Other results related to this paper were proved in [2].

REFERENCES

[1] J. Mycielski, *Remarks on invariant measures in metric spaces*, Colloquium Mathematicum 32 (1974), p. 105-112.

[2] — *A conjecture of Ulam on the invariance of measure in Hilbert's cube*, Studia Mathematica 60 (1977), p. 1-10.

[3] C. A. Rogers, *Hausdorff measures*, Cambridge 1970.

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