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ON COMPACT ONE-TO-ONE CONTINUOUS IMAGES OF THE REAL LINE

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A theorem of Lelek and McAuley [2] shows that every locally compact, locally connected Hausdorff space X, which is the image of the real line $\boldsymbol{R}$ under a one-to-one, continuous function, must be homeomorphic to one of the following simple spaces: (1) an open interval; (2) a figure eight; (3) a dumb-bell; (4) a letter theta; (5) a noose.

To prove this result, they prove the following equivalent theorem, concerning the "endpoints" of X:

If f is a one-to-one, continuous function from $\boldsymbol{R}$ onto a locally compact, locally eonnected Hausdorff space X, and if we define

$$a(f) = \bigcap_{n=1}^{\infty} \mathrm{cl}\{f(t) | t \leqslant -n\}, \qquad \omega(f) = \bigcap_{n=1}^{\infty} \mathrm{cl}\{f(t) | t \geqslant n\},$$

then either $a(f) = \square$ or $\lim\limits_{t\to-\infty} f(t)$ exists; and either $\omega(f) = \square$ or $\lim\limits_{t\to\infty} f(t)$ exists.

If we drop the assumption that X be locally connected, and assume instead that X is compact, we can prove the following analogous result:

THEOREM 1. *Let f be a one-to-one, continuous function from $\boldsymbol{R}$ onto a compact Hausdorff space X, and let $a(f)$ and $\omega(f)$ be defined as above. Then exactly one of the following five conditions occurs*:

(a) *There exist compact intervals I^- and I^+ included in $\boldsymbol{R}$ such that*

$$a(f) = \{f(t) | t \in I^-\}, \qquad \omega(f) = \{f(t) | t \in I^+\}.$$

(b) *There exist a number $\tau \in \boldsymbol{R}$ and a compact interval $I^- \subseteq (-\infty, \tau]$ such that*

$$a(f) = \{f(t) | t \in I^-\}, \qquad \omega(f) = \{f(t) | t \in (-\infty, \tau]\}.$$

(c) *There exist a number $\tau \in \boldsymbol{R}$ and compact intervals I^- and J such that*

$$I^- \subseteq J \subseteq (\tau, \infty), \ a(f) = \{f(t) | t \in I^-\} \text{ and } \omega(f) = \{f(t) | t \in J \cup (-\infty, \tau]\}.$$

(d) *There exist a number* $\tau \in \boldsymbol{R}$ *and a compact interval* $I^+ \subseteq [\tau, \infty)$ *such that*

$$\omega(f) = \{f(t) | t \in I^+\}, \qquad \alpha(f) = \{f(t) | t \in [\tau, \infty)\}.$$

(e) *There exist a number* $\tau \in \boldsymbol{R}$ *and compact intervals* I^+ *and* J *such that*

$$I^+ \subseteq J \subseteq (-\infty, \tau), \ \omega(f) = \{f(t) | t \in I^+\} \text{ and } \alpha(f) = \{f(t) | t \in J \cup [\tau, \infty)\}.$$

LEMMA 1. *Let* f *and* X *be as in the statement of Theorem* 1. *Then* $\alpha(f)$ *and* $\omega(f)$ *are non-empty, compact, and connected.*

Proof. Clear.

LEMMA 2. *Let* f *be a one-to-one continuous function from* $\boldsymbol{R}$ *onto a locally compact Hausdorff space* X.

(a) *There exists a decreasing sequence* $\{t_n\}$ *of real numbers such that*

$$t_n \to -\infty \text{ as } n \to \infty, \text{ and } f(t_n) \notin \alpha(f) \text{ for each } n = 1, 2, \ldots$$

(b) *There exists an increasing sequence* $\{t_n\}$ *of real numbers such that*

$$t_n \to \infty \text{ as } n \to \infty, \text{ and } f(t_n) \notin \omega(f) \text{ for each } n = 1, 2, \ldots$$

Proof. It suffices to prove (a). We remark that $\alpha(f)$, being a closed subset of X, must be locally compact. For each $n = 1, 2, \ldots$ let

$$A_n = \{f(t) \in \alpha(f) | -n \leqslant t \leqslant n\}.$$

Then A_n is closed for each $n = 1, 2, \ldots,$ and $\alpha(f)$ is the union of the sets A_n. Therefore, by the Baire Category Theorem, we can choose an integer N, a point $x \in A_N$, and a neighborhood U of x, such that

$$U \cap \alpha(f) \subseteq A_N.$$

Since $x \in \alpha(f)$, we can choose a decreasing sequence $\{t_n\}$ of real numbers such that $t_n < -N$ for every $n = 1, 2, \ldots, t_n \to -\infty$ as $n \to \infty$, and $f(t_n) \in U$ for every $n = 1, 2, \ldots$ Clearly, since f is one-to-one, we must have

$$f(t_n) \notin \alpha(f) \quad \text{for each } n = 1, 2, \ldots$$

LEMMA 3. *Let* f *and* X *be as in the statement of Theorem* 1.

(a) *Either there exists* $\tau \in \boldsymbol{R}$ *such that* $f(t) \in \alpha(f)$ *for all* $t \geqslant \tau$, *or there exists a compact interval* $I^- \subseteq \boldsymbol{R}$ *such that*

$$\alpha(f) = \{f(t) | t \in I^-\}.$$

(b) *Either there exists* $\tau \in \boldsymbol{R}$ *such that* $f(t) \in \omega(f)$ *for all* $t \leqslant \tau$, *or there exists a compact interval* $I^+ \subseteq \boldsymbol{R}$ *such that*

$$\omega(f) = \{f(t) | t \in I^+\}.$$

Proof. It suffices to prove (a). If we cannot choose $\tau \in \boldsymbol{R}$ such that $f(t) \in a(f)$ for all $t \geqslant \tau$, then choose an increasing sequence $\{t_n\}$ $(n = 1, 2, \ldots)$ such that $t_n \to \infty$ as $n \to \infty$, and $f(t_n) \notin a(f)$ for all $n = 1, 2, \ldots$

Using Lemma 2, choose $\{t_n\}$ $(n = 0, -1, -2, \ldots)$ such that t_n decreases to $-\infty$ as $n \to -\infty$, and $f(t_n) \notin a(f)$ for all $n = 0, -1, -2, \ldots$ Then

$$a(f) = \bigcup_{n=-\infty}^{\infty} a(f) \cap f([t_n, t_{n+1}]),$$

and since this union is a disjoint one, Sierpiński's Theorem (see Kuratowski [1], p. 173) implies that $a(f) \subseteq f([t_n, t_{n+1}])$ for some integer n.

From this and Lemma 1, we deduce easily that there is a compact interval $I^- \subseteq \boldsymbol{R}$ such that

$$a(f) = \{f(t) | t \in I^-\}.$$

LEMMA 4. *Let f and X be as in the statement of Theorem* 1.
(a) *Suppose we can find a number $\sigma \in \boldsymbol{R}$ such that*

$$f(s) \in \omega(f) \quad \textit{for each } s \leqslant \sigma.$$

Then exactly one of the conditions (b), (c) *of Theorem* 1 *holds.*
(b) *Suppose we can find a number $\sigma \in \boldsymbol{R}$ such that*

$$f(s) \in a(f) \quad \textit{for each } s \geqslant \sigma.$$

Then exactly one of the conditions (d), (e) *of Theorem* 1 *holds.*

Proof. Again it suffices to prove (a). Suppose $\sigma \in \boldsymbol{R}$, and $f(s) \in \omega(f)$ for all $s \leqslant \sigma$. Since $\omega(f)$ is compact, we clearly have $a(f) \subseteq \omega(f)$, and we deduce from Lemmas 2 and 3 that there exists a compact interval $I^- \subseteq \boldsymbol{R}$ such that

$$a(f) = \{f(t) | t \in I^-\}.$$

Let $\Omega = \{t \in \boldsymbol{R} | f(t) \in \omega(f)\}$. Ω is a closed subset of $\boldsymbol{R}$, both I^- and $(-\infty, \sigma]$ are contained in Ω, and there are arbitrarily large real numbers which do not belong to Ω. Let $(-\infty, \tau]$ be the component of Ω which contains $(-\infty, \sigma]$, and let J be the component of Ω which contains I^-. It is easy to see that either $J = (-\infty, \tau]$, or J is a compact interval $[a, b]$ disjoint from $(-\infty, \tau]$.

We claim that $\Omega \setminus \big((-\infty, \tau] \cup J\big)$ can be written as a countable disjoint union of compact subsets. We shall prove this only in the case $J = [a, b]$, as the other case is similar (and easier).

Choose sequences $\{s_n\}$ and $\{t_n\}$ $(n = 0, \pm 1, \pm 2, \ldots)$ in $\boldsymbol{R}\setminus\Omega$ such that s_n decreases to τ as $n \to -\infty$, s_n increases to a as $n \to \infty$, t_n decreases to b as $n \to -\infty$, and t_n increases to ∞ as $n \to \infty$. Then the union

$$\bigcup_{n=-\infty}^{\infty} ([s_n, s_{n+1}] \cap \Omega) \cup \bigcup_{n=-\infty}^{\infty} ([t_n, t_{n+1}] \cap \Omega)$$

exhibits $\Omega\setminus\big((-\infty, \tau] \cup J\big)$ as a countable disjoint union of compact subsets.

Thus we can choose a countable, pairwise-disjoint family $\{A_1, A_2, \ldots\}$ of compact sets, whose union is $\Omega\setminus\big((-\infty, \tau] \cup J\big)$. Now $f\big((-\infty, \tau] \cup J\big)$ is clearly a closed subset of $\omega(f)$, and therefore, since

$$\omega(f) = f\big((-\infty, \tau] \cup J\big) \cup \bigcup_{n=1}^{\infty} f(A_n),$$

it follows from Sierpiński's Theorem that $A_n = \square$ for every $n = 1, 2, \ldots$ This shows that $\Omega = (-\infty, \tau] \cup J$, and the Lemma follows easily.

Proof of Theorem 1. Theorem 1 is an elementary consequence of the above lemmas.

Definition. Let f be a one-to-one, continuous function from $\boldsymbol{R}$ onto a compact Hausdorff space X. Then a point $x \in X$ is said to be a *simple endpoint of* f if either $\alpha(f) = \{x\}$ or $\omega(f) = \{x\}$.

LEMMA 5. *Let f and g be continuous, one-to-one functions from $\boldsymbol{R}$ onto a compact Hausdorff space X, and let $x \in X$.*

(a) *$x \in \alpha(f) \cup \omega(f)$ iff $x \in \alpha(g) \cup \omega(g)$.*

(b) *x is a simple endpoint of f iff x is a simple endpoint of g.*

Proof. This lemma may be proved easily, by examining the nature of small compact neighborhoods of x.

THEOREM 2. *Let f and g be continuous one-to-one functions from $\boldsymbol{R}$ onto a compact Hausdorff space X, and let $\varphi(t) = g^{-1}\big(f(t)\big)$ for all t in $\boldsymbol{R}$.*

If $t_0 \in \boldsymbol{R}$ and φ is not continuous on the left [right] at t_0, then $f(t_0)$ is a simple endpoint of f, and as t increases [decreases] to t_0, we have either $\varphi(t) \to \infty$ or $\varphi(t) \to -\infty$.

Proof. This theorem can be proved easily, by an application of Sierpiński's Theorem to a space of the form $f([t_0 - \delta, t_0])$. We omit the details.

Using Theorem 2, we can partition the family $\mathfrak{F}$ of compact, one-to-one, continuous images of $\boldsymbol{R}$ by writing $X \sim Y$ whenever there exist one-to-one, continuous functions f and g from $\boldsymbol{R}$ onto X and Y respectively, such that

$$f^{-1}\big(\alpha(f)\big) = g^{-1}\big(\alpha(g)\big) \quad \text{and} \quad f^{-1}\big(\omega(f)\big) = g^{-1}\big(\omega(g)\big),$$

and then noting that $\sim$ is an equivalence relation in $\mathfrak{F}$.

It can be seen that $\sim$ partitions $\mathfrak{F}$ into the 28 equivalence classes which are illustrated in the figures, and that no two spaces in different classes can be homeomorphic. Classes 1, 2 and 3 are composed of the

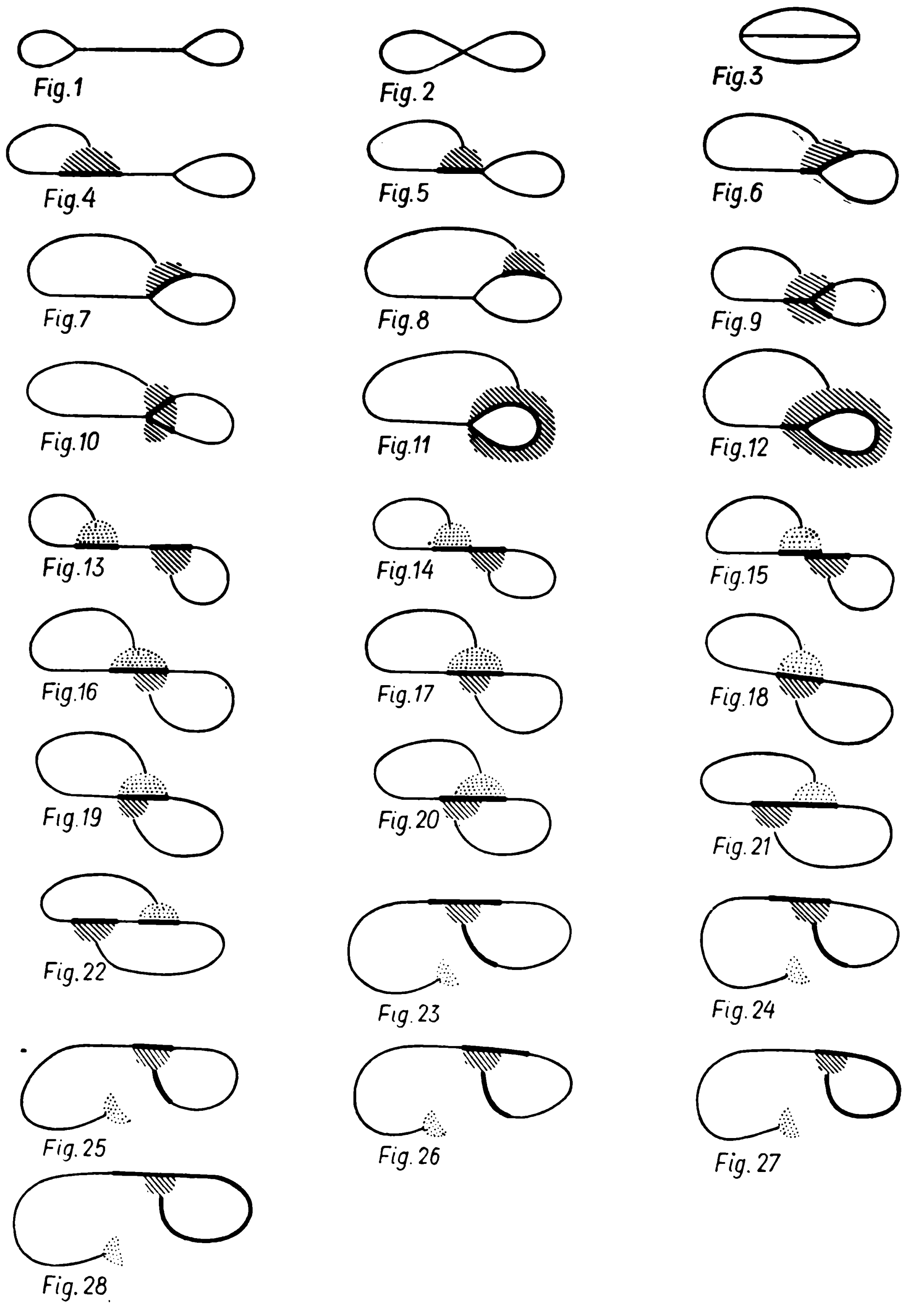

Fig. 1 Fig. 2 Fig. 3 Fig. 4 Fig. 5 Fig. 6 Fig. 7 Fig. 8 Fig. 9 Fig. 10 Fig. 11 Fig. 12 Fig. 13 Fig. 14 Fig. 15 Fig. 16 Fig. 17 Fig. 18 Fig. 19 Fig. 20 Fig. 21 Fig. 22 Fig. 23 Fig. 24 Fig. 25 Fig. 26 Fig. 27 Fig. 28

locally connected members of $\mathfrak{F}$, and up to homeomorphism, each of these classes has only one member. It is not hard to show that each of the other classes has uncountably many non-homeomorphic members.

REFERENCES

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