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FINITE BOLYAI-LOBATCHEVSKY k-SPACES

BY

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Suppose k, n, and m are integers greater than 1 and S is a set. Further, suppose S_i is a class of non-empty subsets of S (for $i = 0, 1, \ldots, k-1$). We will refer to an element of S_i as an i-space. (S_0, S_1, ..., S_{k-1}) is a *finite Bolyai-Lobatchevsky* (B-L) *k-space* if the following axioms hold:

1. $S_0 = \{\{a\} \mid a \in S\}$,
2. $S_1 \neq \emptyset$.

DEFINITIONS. A is an *i-space* of, or is on, a j-space $B \Leftrightarrow A \subset B$ or $B \subset A$. P is a *point* $\Leftrightarrow P \in S_0$. l is a *line* $\Leftrightarrow l \in S_1$. a is a *plane* $\Leftrightarrow a \in S_2$.

3. l is a line $\Rightarrow \exists$ exactly n points, each of which is on l.
4. $l \in S_i \Rightarrow \exists P \in S_0 \ni P \cap l = \emptyset$ $(i \leqslant k-1)$.
5. $l \in S_i$ $(i < k-1)$, $P \in S_0, P \cap l = \emptyset \Rightarrow \exists$ unique $l' \in S_{i+1} \ni l \subset l'$ and $P \subset l'$ (l' may be called lP or Pl; it may be said that l' is *determined* by l and P).
6. P and Q are distinct points of an i-space $A \Rightarrow PQ \subset A$.
7. $l \in S_{k-1}, f \in S_{k-1} \Rightarrow l \cap f = \emptyset, l \cap f = l$, or $l \cap f \in S_{k-2}$.

DEFINITION. Let l and f be i-spaces, each on some $(i+1)$-space g. Then l is *parallel* to $f \Leftrightarrow l \cap f = \emptyset$.

8. If P is a point not on a given line l, then there are exactly m lines on P (and on Pl) parallel to l.

It is the primary purpose of this paper to show that no finite B-L k-space exists for $k > 3$. We will assume that such a space exists, and arrive at a contradiction. Several theorems applicable to the case $k = 3$, are proved in the process.

THEOREMS FOR THE CASES $k > 3$

THEOREM. *A plane is determined by any two lines which have exactly one point in common (whose intersection is a single point).*

Proof. Each line has n points on it, $n > 1$, so there is a point on one line which is not on the other. Such a point and the line not containing it determine a plane. The fact that any two planes so determined are identical is a simple exercise.

THEOREM. *There are $n+m$ lines of a plane on any point of that plane.*

Proof. Given a point P of plane π, $\exists$ a line l not on P. There are n lines on P (and on π) intersecting l, and m lines on p not intersecting l.

THEOREM. *There are $t_2 = (n+m)(n-1)+1$ points on a plane.*

Proof. Let P be a point of plane π. There are $n+m$ lines of π on P and $n-1$ points (other than P) on each such line. The number of points (not P) on lines on P is thus $(n+m)(n-1)$. That t_2 is actually a count of points of π is then a simple exercise, since any point of π is on a line of π on P. Several counting results follow, with a short justification:

1. The number of lines on a plane, q_2, is $n+m/nt_2$. This is a result of the relationship $q_2 n = (n+m)t_2$. This equation is obtained by counting all possible ordered pairs (P_i, l_j) where P_i is a point on the line l_j [2]. There are t_2 points, each with $n+m$ lines on it. Thus there are $t_2(n+m)$ such ordered pairs. But there are q_2 lines, each with n points on it, giving $q_2 n$ ordered pairs. These countings are obviously exhaustive and involve no repetition, so they must yield the same result.

From the equation $q_2 n = (n+m)t_2$, a necessary condition for the existence of a finite B-L plane has been obtained [1]:

$$q_2 n = (n+m)t_2 = nt_2+mt_2 = nt_2+m[(n+m)(n-1)+1],$$

$$q_2 n = nt_2+mn(n-1)+m^2(n-1)+m,$$

$$q_2 n = nt_2+mn^2-mn+m^2n-(m^2-m).$$

Since n is a factor of each term except (m^2-m), and all terms are integers, it is obvious that $n \mid m(m-1)$. This, then is a necessary condition for the existence of the plane.

2. The number of lines on a plane parallel to a given line of that plane is $i = q_2-n(n+m-1)-1$. Suppose l is the given line. There are q_2-1 lines (other than l) on the given plane. At each of the n points of l, $n+m-1$ of the lines (not l) intersect l. Thus $n(n+m-1)$ lines other than l intersect l. Lines parallel to l are thus i in number.

3. There are $i^* = i-(n-1)m$ lines parallel to each of two intersecting (distinct) lines and in the plane determined by those two lines. Let h and j be the lines, with intersection the point P. There are i lines (on the plane of h and j) parallel to j. On each point of h (not P) are exactly m of these parallels. There are $n-1$ points of h other than P, or $m(n-1)$ of the i lines parallel to j are not parallel to h. This leaves $i-m(n-1)$ lines parallel to both lines j and h (I omit, of course, certain details such as distinctness in these abbreviated proofs). Manipulation yields $i^* = m(m-1)(n-1)/n$.

4. $t_3 = (n+m)(t_2-n)+n$ is the number of points on a 3-space. Points of a 3-space are all on the planes which are on a given line of that 3-space. There are $n+m$ planes on a line of the 3-space. To prove this, consider a given line l, and P a point of l. Certainly on P there is a plane π containing no other points of l. Then planes on l intersect π in lines on P. There are exactly $n+m$ lines of π on P, hence exactly $n+m$ planes of the 3-space on l. Points not on l are t_2-n in number for each such plane, so there are $(t_2-n)(n+m)$ points not on l in the 3-space. Adding the n points of l gives the formula for t_3.

5. Let π be a plane and P a point not on π. Then the number of planes on P (and on the 3-space $P\pi$) parallel to π is

$$m^2-i^* = m^2-m(m-1)\left(\frac{n-1}{n}\right).$$

The proof here is somewhat longer. Let l and j be two distinct lines of π intersecting at point Q. On P and Pj are m lines parallel to j, say $j_1, j_2, \ldots, j_m$. Also, on P and Pl are m lines parallel to l: $l_1, l_2, \ldots, l_m$. Make the convention that A_{rs} is the plane determined by l_r and j_s. There are m^2 planes of this type, since there are exactly m^2 pairs (r, s) and, by a simple argument, distinct planes are associated with distinct pairs. Any plane on P (and on $P\pi$) parallel to π must be an A_{rs} for some r and s. Indeed, a plane α on P parallel to π has a trace on Pj (a line of intersection with P_j), say u. u is parallel to j, because if R is a point of u and of j, then R is a point of π and of α. But α is parallel to π. Therefore u is parallel to j. That is, $u = j_s$ for some s. Similarly, the trace of α on Pl is l_r for some r.

Thus, the number of planes on P parallel to π is the total number of A_{rs} planes (m^2) diminished by the number of A_{rs} planes which intersect π.

Let us consider a plane A_{hg} which intersects π. Suppose β is the trace it has on π. If β intersects j, say in point S, then S is on A_{hg} and on P_j. Then P and S are points of A_{hg} and of Pj, which implies that PS is on each of these planes, and must accordingly be their (unique) line of intersection. Now the intersection of A_{hg} and Pj is j_g, a line parallel to j. But PS is not parallel to j since S is a point of j. Thus, the assumption that β intersects j is incorrect. It is similarly proved that β does not intersect l. The aim of all this is to show that β is a line parallel to each of two intersecting lines (l and j). It is then fairly clear that there is a one-to-one correspondence of A_{rs} planes which intersect π and lines of π parallel to each of l and j. There are i^* of these lines, and accordingly $i*$ of the A_{rs} planes which intersect π. We have proved that there are m^2-i^* of the A_{rs} planes which do not intersect π. Only A_{rs} planes (of all planes on P) can be parallel to π, so the number of planes on P parallel to π is $m^2 - i*$. It should be noticed that this number is positive.

6. We are now in a position to show that no finite B-L k-space exists ($k > 3$). We begin with the assumption that such a k-space does exist.

Accordingly, there exists (in some 4-space) a plane π, a point P not on π and the 3-space $P\pi$. There is a point Q not on $P\pi$, and the 3-space $Q\pi$ exists accordingly. $Q\pi \neq P\pi$, so $Q\pi \cap P\pi = \pi$. On Q (and on $Q\pi$) there is at least one plane a parallel to π (from 5, above), and a is thus disjoint from $P\pi$. Then for any point S of $P\pi$, Sa is a 3-space (distinct from $P\pi$) having non-empty intersection with $P\pi$. Thus $Sa \cap P\pi$ is a plane S' of $P\pi$ containing S. With a fixed, each point of $P\pi$ thus has associated with it a unique plane of $P\pi$. The set of planes so determined has the interesting property that each point of $P\pi$ is on exactly one of them. Let us suppose that some point T is on two of these planes, A and B. Then A is easily seen to be the (unique) intersection of Ta and $P\pi$, as is B. Thus $A = B$.

All of this is to demonstrate that the points of $P\pi$ must be arranged on a set of pairwise disjoint planes. There are t_3 points thusly arranged on planes (each containing t_2 points); so, for some integer s, $st_2 = t_3$. Then $st_2 = t_3 = (t_2 - n)(n+m)+n$,

$$st_2 = (n+m)t_2 + n - n(n+m). \tag{1}$$

But $t_2 = (n+m)(n-1)+1 = (n+m)n - n - m + 1$, or

$$n - (n+m)n = -t_2 - m + 1. \tag{2}$$

Substituting (2) in (1), we get

$$st_2 - (n+m)t_2 = -t_2 - (m-1);$$

t_2 is a factor of all terms except $(m-1)$, so $t_2 \mid (m-1)$. This is impossible, because $t_2 > m-1$ and $m-1 > 0$. This contradiction proves that *no finite B-L k-space exists if* $k > 3$.

7. Whether a finite B-L 3-space exists or not is an open question as far as the author has been able to determine. Two necessary conditions have been found, however. They are:

$n \mid (m-1)$,

$t_2 \mid m(m-1)$.

The first is proved by a construction similar to that used to prove that all points of a 3-space (in a higher space) are on a set of pairwise disjoint planes. With the argument scaled down one "dimension", it can be shown that the points of a plane (in a 3-space) are on a set of pairwise disjoint lines. Thus, for some integer s:

$$sn = t_2 = (n+m)(n-1)+1,$$
$$sn = n(n+m)-n-(m-1);$$

n is a factor of all terms except $m-1$, so $n \mid (m-1)$.

The second is proved by a consideration of the number z, of planes on a point (in a 3-space), the number x of planes in a 3-space, and t_2 and t_3. By an argument similar to that used to determine the number of lines on a plane, the equation

$$\text{(A)} \qquad t_2 x = t_3 z$$

arises. This equation becomes useful once z is known.

To find z, we first determine the number of lines on a point (in a 3-space). The points on a 3-space are obviously all on lines on a given point P. $n-1$ points other than P are on each of these lines. If there are s lines on P, then there are $s(n-1)$ points (not P) in the 3-space. Thus there are $s(n-1)+1$ points in all in the 3-space. Therefore $s(n-1)+1 = t_3$. Solving, we obtain $s = (n+m)(n+m-1)+1$.

We use s next to determine z. Let us consider (in a 3-space) a point P and a plane π on P. There are $n+m$ lines on π and on P. There are $\binom{n+m}{2}$ pairs of (distinct) lines on P and on π, and each pair determines the plane π. Any pair of lines on P determines a plane on P. With a little reflection, it becomes obvious that the number of *pairs* of lines on P is the product of the number of pairs of lines on a given plane (on P) and the total number of planes on P. Thus

$$\binom{s}{2} = \binom{n+m}{2} z,$$

regardless of the choice of P.

Solving:

$$\begin{aligned} z &= s(s-1)/(n+m)(n+m-1) = s(s-1)/(s-1) = s \\ &= (n+m)(n+m-1)+1 \\ &= (n+m)(n-1)+1+m(n+m) \\ &= t_2+m(n+m). \end{aligned}$$

We use this expression for z in equation (A):

$$t_2 x = t_3 z = t_3[t_2+m(n+m)],$$
$$t_2 x = t_3 t_2 + t_3 m(n+m),$$
$$t_2 x = t_3 t_2 + [(n+m)(t_2-n)+n]m(n+m)$$

or

$$t_2x = t_3t_2 + (n+m)(t_2-n)(m)(n+m) + nm(n+m),$$
$$t_2x = t_3t_2 + (n+m)^2t_2m - mn(n+m)^2 + mn(n+m),$$

or

$$\text{(B)} \qquad t_2x - t_3t_2 - (n+m)^2t_2m = -m(n+m)(n)(n+m-1).$$

It is easy to verify that $n(n+m-1) = t_2+m-1$.

Substituting in the right hand member of (B) yields:

$$t_2x - t_3t_2 - (n+m)^2t_2m = -m(n+m)(t_2+m-1)$$

or

$$t_2x - t_3t_2 - (n+m)^2t_2m = -m(n+m)t_2 - m(m-1)(n+m).$$

Obviously t_2 divides $m(m-1)(n+m)$.

Then for some non-zero integer r,

$$\text{(C)} \qquad m(m-1)(n+m) = rt_2 = r[(n+m)(n-1)+1],$$
$$m(m-1)(n+m) = r(n+m)(n-1)+r.$$

Hence $(n+m) \mid r$, and $r = d(n+m)$ for some non-zero integer d. Substituting in (C) $m(m-1)(n+m) = d(n+m)t_2$, we get

$$m(m-1) = dt_2.$$

From this, it follows that $t_2 \mid m(m-1)$.

REFERENCES

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