COLLOQUIUM MATHEMATICUM

VOL. LVIII 1989 FASC. 1

# ATOMS OF CHARACTERISTIC MEASURES

BY

K. URBANIK (WROCŁAW)

In this paper we adopt the definitions and notation given in [1] and [2]. In particular, $P$ will denote the space of all Borel probability measures defined on the positive half-line $[0, \infty)$. The space $P$ is endowed with the topology of weak convergence. For any $a \in (0, \infty)$, $T_a$ will denote the scale change

$$(T_a \mu)(E) = \mu(a^{-1} E) \quad (\mu \in P).$$

Further, $\delta_c$ will denote the probability measure concentrated at the point $c$. Two measures $\mu$ and $\nu$ from $P$ are said to be *similar*, in symbols $\mu \sim \nu$, if $\mu = T_a \nu$ for a certain $a \in (0, \infty)$. A continuous commutative and associative $P$-valued binary operation $\circ$ on $P$ is called a *generalized convolution* if it is distributive with respect to the convex combinations of measures and the operations $T_a$ $(a > 0)$, $\delta_0$ is its unit element, and an analogue of the law of large numbers is fulfilled:

$$T_{c_n} \delta_1^{\circ n} \to \gamma \neq \delta_0$$

for a choice of a norming sequence $c_n$ of positive numbers. The power $\delta_1^{\circ n}$ is taken here in the sense of the operation $\circ$. The limit measure $\gamma$ is called a *characteristic measure* of the generalized convolution in question. It is known by Proposition 4.4 in [2] that the characteristic measure is uniquely determined up to the equivalence relation $\sim$ . Moreover, by Proposition 4.5 in [2] there exists a constant $\varkappa = \varkappa(\circ)$ belonging to $(0, \infty]$ and called the *characteristic exponent* of the generalized convolution $\circ$ such that

$$T_a \gamma \circ T_b \gamma = T_{g_\varkappa(a,b)} \gamma \tag{1}$$

for any pair $a, b \in (0, \infty)$, where

$$g_\varkappa(a, b) = (a^\varkappa + b^\varkappa)^{1/\varkappa} \quad \text{if } \varkappa \in (0, \infty)$$

and

$$g_\infty(a, b) = \max(a, b).$$

The aim of this note is to investigate the atoms of characteristic measures. Given $\mu \in P$ we denote by $A(\mu)$ the set of all atoms of $\mu$, i.e.,

$$A(\mu) = \{a\colon \mu(\{a\}) > 0\}.$$

For any $b \in (0, \infty)$ we have the formula

$$A(T_b \mu) = bA(\mu). \tag{2}$$

Moreover, by Lemma 2.2 in [2], the relation

$$0 \notin A(\gamma) \tag{3}$$

is true. Taking into account formula (2) and choosing an appropriate equivalent version of $\gamma$ if necessary, we assume in this paper without loss of generality that the condition

$$1 \in A(\gamma) \quad \text{whenever } A(\gamma) \neq \emptyset \tag{4}$$

is fulfilled.

Given $p \in (0, \infty]$ and two independent non-negative random variables $X$ and $Y$ with probability distributions $\mu$ and $\nu$, respectively, we denote by $\mu *_p \nu$ the probability distribution of the random variable $g_p(X, Y)$. The operation $*_p$ is a generalized convolution with the characteristic exponent $p$ and the characteristic measure $\delta_1$. Thus $A(\gamma) \neq \emptyset$ for all generalized convolutions $*_p$ ($p \in (0, \infty]$). The converse implication is also true.

THEOREM. *Let $\circ$ be a generalized convolution for which the characteristic measure has at least one atom. Then $\circ = *_p$, where $p = \varkappa(\circ)$.*

Before proving the Theorem we shall prove some lemmas.

LEMMA 1. *If $A(\gamma) \neq \emptyset$, then for every $a, b \in A(\gamma)$ and every $x, y \in (0, \infty)$ the inclusion*

$$A(\delta_{ax} \circ \delta_{by}) \subset g_\varkappa(x, y) A(\gamma)$$

*is true.*

Proof. Let $a, b \in A(\gamma)$. Then the measure $\gamma$ can be written in the form

$$\gamma = q\delta_a + r\delta_b + s\lambda,$$

where $q, r > 0$, $s \geqslant 0$, $q + r + s = 1$ and $\lambda \in P$. Hence it follows that for any $x, y \in (0, \infty)$

$$T_x \gamma \circ T_y \gamma = qr \delta_{ax} \circ \delta_{by} + (1 - qr)\varrho$$

with a certain $\varrho \in P$. Thus

$$A(\delta_{ax} \circ \delta_{by}) \subset A(T_x \gamma \circ T_y \gamma).$$

Our assertion is now a direct consequence of formulae (1) and (2).

If $\varkappa(\circ) < \infty$ and $A(\gamma) \neq \emptyset$, then by $K(\gamma)$ we shall denote the denumerable number field generated by $\varkappa(\circ)$-th powers of elements of $A(\gamma)$. In what follows $\operatorname{card} B$ will denote the cardinality of the set $B$.

LEMMA 2. *Suppose that $\varkappa(\circ) < \infty$ and* $\operatorname{card} A(\gamma) > 1$. *Then*

$$A(\delta_u \circ \delta_v) = \emptyset$$

*for any pair $u, v \in (0, \infty)$ for which the numbers $u^\varkappa$ and $v^\varkappa$ are linearly independent over the field $K(\gamma)$.*

Proof. Suppose that the pair $u$, $v$ fulfils the condition of the lemma. By (4) we have the relation $1 \in A(\gamma)$. Since $\operatorname{card} A(\gamma) > 1$, we can choose a number $c \in A(\gamma)$ such that $c \neq 1$. Moreover, by (3), we have the inequality $c > 0$. Setting $a = b = 1$, $x = u$, $y = v$ and $a = 1$, $b = c$, $x = u$, $y = c^{-1}v$ into the assertion of Lemma 1 we obtain the inclusions

$$A(\delta_u \circ \delta_v) \subset g_\varkappa(u, v) A(\gamma)$$

and

$$A(\delta_u \circ \delta_v) \subset g_\varkappa(u, c^{-1}v) A(\gamma).$$

Consequently, the inequality $A(\delta_u \circ \delta_v) \neq \emptyset$ and formula (3) would imply the existence of a pair of positive numbers $a$, $b$ in $A(\gamma)$ such that

$$(u^\varkappa + v^\varkappa) a^\varkappa = (u^\varkappa + c^{-\varkappa} v^\varkappa) b^\varkappa.$$

But, by the linear independence of the numbers $u^\varkappa$ and $v^\varkappa$ over the field $K(\gamma)$, the above equality is impossible, which completes the proof.

LEMMA 3. *Suppose that* $\varkappa(\circ) < \infty$ *and* $A(\gamma) \neq \emptyset$. *If* $0 \notin A(\mu)$ *and* $A(\nu) = \emptyset$, *then*

$$A(\mu \circ \nu) = \emptyset.$$

Proof. Suppose the contrary, i.e.,

$$A(\mu \circ \nu) \neq \emptyset.$$

Let $a \in A(\mu \circ \nu)$. Then, by Lemma 1.2 in [2],

$$(\mu \circ \nu)(\{a\}) = \int_0^\infty \int_0^\infty (\delta_x \circ \delta_y)(\{a\}) \nu(dx) \mu(dy).$$

Since $0 \notin A(\mu)$, we can find a positive number $w$ satisfying the condition

$$\int_0^\infty (\delta_x \circ \delta_w)(\{a\}) \nu(dx) > 0,$$

which, by the assumption $A(\nu) = \emptyset$, yields that the set

$$\{x \colon (\delta_x \circ \delta_w)(\{a\}) > 0\}$$

is non-denumerable. Consequently, we can find a pair $u$, $v$ of positive numbers such that

$$(\delta_u \circ \delta_w)(\{a\}) > 0, \quad (\delta_v \circ \delta_w)(\{a\}) > 0,$$

and the numbers $u^\varkappa$, $v^\varkappa$, $w^\varkappa$ are linearly independent over the field $K(\gamma)$. Applying Lemma 1 we get the relation

$$a \in A(\delta_u \circ \delta_w) \cap A(\delta_v \circ \delta_w) \subset g_\varkappa(u, w) A(\gamma) \cap g_\varkappa(v, w) A(\gamma),$$

which, by (3), implies the existence of a pair of positive numbers $b, c \in A(\gamma)$ such that

$$(u^{\varkappa}+w^{\varkappa})b^{\varkappa} = (v^{\varkappa}+w^{\varkappa})c^{\varkappa}.$$

But this contradicts the linear independence of the triple $u^{\varkappa}$, $v^{\varkappa}$, $w^{\varkappa}$ over the field $K(\gamma)$. The lemma is thus proved.

Proof of the Theorem. By Lemma 2.1 in [2] the equality $\varkappa(\circ) = \infty$ yields $\circ = *_{\infty}$. Consequently, we may restrict ourselves to the case $\varkappa(\circ) < \infty$. It is clear that the characteristic measure $\gamma$ can be written in the form

$$\gamma = q\varrho+(1-q)\lambda, \tag{5}$$

where

$$0 < q \leqslant 1, \tag{6}$$

$$\varrho = \sum_{a\in A(\gamma)} q_a\delta_a, \tag{7}$$

$$q_a > 0 \quad \text{for } a\in A(\gamma),$$

$$\sum_{a\in A(\gamma)} q_a = 1 \quad \text{and} \quad A(\lambda) = \emptyset.$$

Further, by (3), the condition $0 \notin A(\varrho)$ is fulfilled. Consequently, by Lemma 3,

$$A(\varrho\circ\lambda) = A(\lambda\circ\lambda) = \emptyset,$$

which yields the formula

$$\gamma\circ\gamma = q^2\varrho\circ\varrho+(1-q^2)\,\nu, \tag{8}$$

where $A(\nu) = \emptyset$. On the other hand, setting $c = g_{\varkappa}(1, 1)$ we have, by (1) and (5),

$$\gamma\circ\gamma = T_c\gamma = q\,T_c\varrho+(1-q)\,T_c\lambda. \tag{9}$$

Taking into account that the measure $\varrho$ is purely atomic and comparing the right-hand sides of (8) and (9) we get the inequality $q \leqslant q^2$, which, by (6), implies the formula $q = 1$. Thus, by (5) and (7), the characteristic measure $\gamma$ is of the form

$$\gamma = \sum_{a\in A(\gamma)} q_a\delta_a. \tag{10}$$

Consequently, for any pair $x$, $y$ of positive numbers we have the formula

$$T_x\gamma\circ T_y\gamma = \sum_{a,b\in A(\gamma)} q_a q_b \delta_{ax}\circ\delta_{by},$$

which, by formulae (1) and (2), yields

$$g_{\varkappa}(x,y)A(\gamma) = \bigcup_{a,b\in A(\gamma)} A(\delta_{ax}\circ\delta_{by}). \tag{11}$$

Taking a pair $x, y$ of positive numbers for which $x^{\varkappa}$ and $y^{\varkappa}$ are linearly independent over the field $K(\gamma)$, we conclude, by (3), that for every $a, b \in A(\gamma)$ the numbers $a^{\varkappa}u^{\varkappa}$ and $b^{\varkappa}v^{\varkappa}$ are also linearly independent over $K(\gamma)$. Assume that the set $A(\gamma)$ contains at least two elements. Then, by Lemma 2,

$$A(\delta_{ax} \circ \delta_{by}) = \emptyset \quad \text{for all } a, b \in A(\gamma),$$

which, by (11), yields $A(\gamma) = \emptyset$. But this contradicts the assumption. Consequently, $A(\gamma)$ is a one-point set, which, by assumption (4), gives the formula $A(\gamma) = \{1\}$. Thus, by (10), $\gamma = \delta_1$ and, finally, by (1),

$$\delta_x \circ \delta_y = \delta_{g_{\varkappa}(x,y)} \quad \text{for all } x, y \in (0, \infty).$$

Hence the equality $\circ = *_{\varkappa}$ follows, which completes the proof.

## REFERENCES

[1] K. Urbanik, *Generalized convolutions*, Studia Math. 23 (1964), pp. 217–245.
[2] – *Generalized convolutions. IV*, ibidem 53 (1986), pp. 57–95.

INSTITUTE OF MATHEMATICS
WROCŁAW UNIVERSITY
WROCŁAW, POLAND

*Reçu par la Rédaction le 10.10.1988*