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A CLASS OF I_0-SETS

BY

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1. Introduction. A set $E \subset R$, the real numbers, is an *I_0-set* ([1]) if every bounded complex-valued function on E can be extended to an almost periodic function on R. It is a classical result of Mycielski [6] and Strzelecki [8] that any lacunary sequence, $0 < q_1 < q_2 \ldots$ where $q = \inf\{q_{j+1}/q_j\colon j = 1, 2, \ldots\} > 1$, is an I_0-set. Méla [5] produced other examples of I_0-sets by summing pairs of elements from a sufficiently gapped lacunary sequence to form what we describe in the next paragraph as a restricted blocked set. In this investigation a more general class called *blocked sets* is examined. These are shown to be I_0-sets when the associated lacunary sequence is sufficiently gapped.

Suppose $\Lambda = \{q_j\}$ is a lacunary sequence. Let $K = \{k_j\}$ be any subsequence of Λ, and let $\Lambda(k_j)$ be any sequence of disjoint subsets of Λ. Define the blocked set $E = \bigcup (k_j + \Lambda(k_j))$, $j \in Z^+$; here $k+S = \{k+s\colon s \in S\}$. When $K \cap \Lambda(k_j) = \emptyset$ for all j, E is called a *restricted blocked set*. The elements of K are called the *translators*, and the elements of $\Lambda(k_j)$ are said to be *associated to* k_j. In general, blocked sets differ significantly from lacunary sets. Indeed, a blocked set such as $\{3^{j^2}+3^k\colon j, k \in Z^+, (j-1)^2 \leqslant k < j^2\}$ is often cited as an example of a Sidon set which is not a finite union of lacunary sets ([2, p. 8], [3, p. 132], [4, p. 25], [7, p. 127]).

The fundamental result of this investigation is the following theorem.

THEOREM 1. *Let $\Lambda = \{q_j\}$ be a lacunary sequence with lacunary ratio $q > 2$. Then any blocked set E formed from Λ is an I_0-set.*

2. Outline of proof of Theorem 1. We will make use of the following result [1, Theorem 1].

PROPOSITION 1. *For a set $E \subset R$ to be an I_0-set, it is sufficient that every function on E taking values 0 and 1 can be extended to an almost periodic function on R.*

Let $t = (t_1, \ldots, t_n)$ be a vector in R^n and let $r \in R$. Define tr

$= (t_1 r, \ldots, t_n r)$, let $\|r\|$ denote the distance from r to the nearest integer, and set $\|t\| = \max\{\|t_i\|: 1 \leqslant i \leqslant n\}$. Given two disjoint sets A, $B \subset R$, we will say that A and B are *separated* (or more precisely, *separated in the Bohr group*) if there exist $n \in Z^+$, $\delta > 0$, and $t \in R^n$ such that $\|tx - ty\| \geqslant \delta$ for all $x \in A$ and all $y \in B$. Such a vector t will be said to *separate* A and B. Proposition 1 immediately implies the next result.

PROPOSITION 2. *For $E \subset R$ to be an I_0-set, it is sufficient that for any partition $E = E_0 \cup E_1$, the sets E_0 and E_1 are separated.*

If $q > 6$, then Méla showed any restricted blocked set E formed from Λ is I_0. His proof consisted of showing that any partition of $E = E_0 \cup E_1$ could be separated using a one-dimensional vector. Our proof of Theorem 1 is based on a subdivision of E into two I_0-sets which are separated from one another. A key tool in this regard is Lemma 9 which is referred to as the "partition lemma". It allows us to separate the various pieces of the two I_0-subsets using one-dimensional vectors. The one-dimensional vectors are produced using two general separating lemmas (3 and 4). These lemmas, which are proved via nested interval arguments, depend on the parameters ν, δ, μ, λ, α which are all functions of the lacunary ratio q of Λ and which, in turn, determine the subdivision parameters ε, r, s.

Given $\varepsilon > 0$, define

$$N = \{k + l \in E: k \in K, l \in \Lambda(k), \text{ and } l/k < \varepsilon\} \quad \text{and} \quad M = E \setminus N.$$

Theorem 1 is then an immediate corollary of the following four propositions.

PROPOSITION 3. *M is an I_0-set.*

PROPOSITION 4. *M and N are separated.*

PROPOSITION 5. *N is an I_0-set.*

PROPOSITION 6. *Suppose that $F = \bigcup_{i=1}^{N} F_i$ and $G = \bigcup_{j=1}^{M} G_j$ are subsets of R. If for each pair of indices (i, j) F_i and G_j are separated, then F and G are separated.*

The proof of Proposition 6 is elementary and it is omitted.

3. Proof of Theorem 1. The first two lemmas are preparatory to the two basic separating lemmas (3 and 4). If $r \in R$, let $r \in [a, b] \bmod 1$ denote the statement $r \in [a, b] + Z = \{t + n: a \leqslant t \leqslant b, n \in Z\}$.

LEMMA 1. *Suppose $0 < y < x$ and define $\eta = x/y$. Let $-1/2 \leqslant \sigma, \tau \leqslant 1/2$ and $0 < \delta < 1/8$. Set*

$$N = \min\{n \in Z^+ \cup \{0\}: n \geqslant [2 - (1 - 8\delta)\eta]/[2(\eta - 1)]\}.$$

Let Q be any closed interval of length $|Q| \geqslant (2N + 3)/(2y)$.

(i) *There is a closed interval* $J \subset Q$ *satisfying* $|J| = \delta/x$ *and such that* $ty \in [\sigma, \sigma+(1/2)] \bmod 1$ *and* $tx \in [\tau-\delta, \tau+\delta] \bmod 1$ *for all* $t \in J$.

(ii) *Let* $0 < v < 1/2$ *and suppose that* δ *satisfies* $(16\delta)/(1+8\delta) \leqslant v$. *If* $\eta \notin (2-v, 2+v)$, *then there is a closed interval* $J \subset Q$ *satisfying the equality* $|J| = \delta/x$ *such that* $ty \in [\sigma, \sigma+(1/2)-4\delta] \bmod 1$ *and* $tx \in [\tau-\delta, \tau+\delta] \bmod 1$ *for all* $t \in J$.

Proof. For $k, l \in Z$ define $J(l, +) = [(l+\tau)/x, (l+\tau+\delta)/x]$, $J(l, -) = [(l+\tau-\delta)/x, (l+\tau)/x]$, and $I(k) = [(k+\sigma)/y, (k+\sigma+\mu)/y]$ where $\mu = 1/2$ in case (i) and $\mu = (1/2)-4\delta$ in case (ii). If $\eta \geqslant 2$, then $|I(k)| \geqslant 1/x$ for each k and $|Q| \geqslant 3/(2y)$ so $J = J(l, +) \subset I(k) \subset Q$ or $J = J(l, -) \subset I(k) \subset Q$ for appropriate $k, l \in Z$. Now suppose $1 < \eta < 2$. Clearly we may assume that Q is a finite interval. Define $k_0 = \min\{k \in Z: I(k) \subset Q\}$ and define $l = \min\{n \in Z: (n+\tau)/x \geqslant (k_0+\sigma)/y\}$. If $I(k_0)$ contains a point of the form $(n+\tau)/x$ where $n \in Z$, then define J to be the closed interval $J(n, +)$ or $J(n, -)$, whichever is contained in Q. Therefore we may assume that $I(k_0)$ contains no point $(n+\tau)/x$ where $n \in Z$. This implies that $[(l+\tau)/x] - [(k_0+\sigma+\mu)/y] > 0$. Define $M \in Z^+$ by

$$M = \min\{m \in Z: m[(1/y)-(1/x)] \geqslant [(l+\tau)/x]-[(k_0+\sigma+\mu)/y]\}.$$

It is easily verified that $M \leqslant N$ and $(l+\tau+M)/x \in I(k_0+M)$. The inequality implies that $|Q| \geqslant (2M+3)/(2y)$, and hence that $I(k_0+M) \subset Q$. Define J to be the closed interval $J(l+M, +)$ or $J(l+M, -)$, whichever is contained in $I(k_0+M)$.

LEMMA 2. *Let* $0 < y < x$ *and define* $\eta = x/y$. *Suppose that* $-1 < a, b, c, d < 1$ *where* $0 < b-a < 1$, $0 < d-c < 1$, *and* $(1-d+c)/(b-a) < \eta$. *Let* $F = [a, b]$ *and* $G = [c, d]$. *Let* Q *be any closed interval satisfying* $|Q| \geqslant (1+b-a)/y$.

(i) *If* $\eta \geqslant (1+d-c)/(b-a)$, *then* Q *contains a closed interval* I *satisfying* $|I| = (d-c)/x$ *such that* $ty \in F \bmod 1$ *and* $tx \in G \bmod 1$ *for all* $t \in I$.

(ii) *If* $(1+d-c)/(b-a) > \eta$, *then* Q *contains a closed interval* I *satisfying* $|I| \geqslant [\eta(b-a)-(1-d+c)]/(2x)$ *such that* $ty \in F \bmod 1$ *and* $tx \in G \bmod 1$ *for all* $t \in I$.

Proof. Omitted.

LEMMA 3. *Let* $0 < v < 1/2$ *and suppose that* $0 < \delta < 1/48$ *is small enough so that* $(16\delta)/(1+8\delta) < v$. *Assume that* μ *is large enough so that* $\mu \geqslant 5/(2\delta)$, *and let* $-1/2 < \zeta, \varrho, \theta, \theta' < 1/2$. *Suppose that* A, B, C, *and* C' *are disjoint, countable subsets of* R^+ *such that* $A \cup B \cup C \cup C' = \{q_j\}$ *has the following properties*:

(1) $q_{j+1}/q_j \in [3/2, 2-v] \cup [2+v, \infty)$;

(2) $q_{j+2}/q_j \geqslant \mu^2$;

(3) *if* $q_j \in C \cup C'$, *then* $q_{j+1}/q_j \geqslant \mu$.

Let Q *be any closed interval satisfying* $|Q| \geqslant 5/(2q_1)$. *Then there is a* $t \in Q$

such that

$$\text{(P)}\qquad \begin{aligned} &tq_j \in [\zeta, \zeta+(1/2)-4\delta] \bmod 1 \quad \textit{for all } q_j \in A,\\ &tq_j \in [\varrho, \varrho+(1/2)-4\delta] \bmod 1 \quad \textit{for all } q_j \in B,\\ &tq_j \in [\theta-\delta, \theta+\delta] \bmod 1 \quad \textit{for all } q_j \in C,\\ &tq_j \in [\theta'-\delta, \theta'+\delta] \bmod 1 \quad \textit{for all } q_j \in C'. \end{aligned}$$

Proof. In Lemma 2 take let us $b-a = d-c = (1/2)-4\delta$ and $x, y \in A \cup B \cup C \cup C'$. Therefore, since $\eta \geqslant 3/2$ and $0 < \delta < 1/48$, the length of the interval I guaranteed by Lemma 2 will satisfy

$$|I| \geqslant [\eta(b-a)-(1-d+c)]/(2x) \geqslant 1/(16x) > \delta/x.$$

Also, it is easy to see that the maximum value of $f(\eta) = [2-(1-8\delta)\eta]/[2(\eta-1)]$ for $\eta \in [3/2, \infty)$ is $f(3/2) = (1+24\delta)/2 \leqslant 3/4 < 1$. Thus, under the hypotheses of Lemma 3, we may apply Lemma 1 with $N = 1$.

Now we will construct inductively a sequence of integers $0 = N(0) < N(1) < N(2) < \ldots$ and a sequence of closed intervals $Q = I(0) \supset I(1) \supset I(2) \supset \ldots$ such that for each positive integer k, for every $t \in I(k)$, and for every positive integer $j \leqslant N(k)$, (P) is valid. Furthermore the sequences will be chosen such that $|I(k)| \geqslant \delta/q_{N(k)}$ and $q_{N(k)+1}/q_{N(k)} \geqslant \mu$ for each $k \geqslant 1$. The number $t \in \bigcap I(k)$ will have the desired properties.

To start the induction, note that $|Q| = |I(0)| \geqslant 5/(2q_1)$. The argument given in cases 1 and 2 of the induction step below shows that there is an integer $N(1) \in \{1, 2\}$ such that $q_{N(1)+1}/q_{N(1)} \geqslant \mu$, and a closed interval $I(1) \subset I(0)$ satisfying $|I(1)| \geqslant \delta/q_{N(1)}$ such that (P) is valid for all $j \leqslant N(1)$ and all $t \in I(1)$.

Suppose that $k \geqslant 1$ and integers $0 = N(0) < \ldots < N(k)$ and closed intervals $Q = I(0) \supset \ldots \supset I(k)$ have been found such that (P) is valid for all $t \in I(k)$ and all $j \leqslant N(k)$, that $|I(k)| \geqslant \delta/q_{N(k)}$, and that $q_{N(k)+1}/q_{N(k)} \geqslant \mu$.

Case 1. $q_{N(k)+2}/q_{N(k)+1} < \mu$. Define $N(k+1) = N(k)+2$ and note that property (2) implies

$$q_{N(k+1)+1}/q_{N(k+1)} = q_{N(k)+3}/q_{N(k)+2} > \mu.$$

Also note that property (3) implies $q_{N(k)+1} \notin C \cup C'$. Therefore $q_{N(k)+1} \in A \cup B$ and either $q_{N(k)+2} \in C \cup C'$ or $q_{N(k)+2} \in A \cup B$. If $q_{N(k)+2} \in C \cup C'$, then in Lemma 1 take $x = q_{N(k)+2}$, $y = q_{N(k)+1}$, $Q = I(k)$, $\sigma = \zeta$ or ϱ depending on whether $q_{N(k)+1} \in A$ or B respectively, and $\tau = \theta$ or θ' depending on whether $q_{N(k)+2} \in C$ or C'. Since $q_{N(k)+1}/q_{N(k)} \geqslant \mu \geqslant 5/(2\delta)$, it follows that

$$|Q| = |I(k)| \geqslant \delta/q_{N(k)} \geqslant 5/2q_{N(k)+1}.$$

Lemma 1 guarantees a closed interval $I(k+1) \subset I(k)$ satisfying $|I(k+1)| \geqslant \delta/q_{N(k)+2} = \delta/q_{N(k+1)}$ such that, for all $t \in I(k+1)$, $tq_{N(k)+2} \in [\tau-\delta, \tau+\delta] \bmod 1$ and $tq_{N(k)+1} \in [\sigma, \sigma+(1/2)-4\delta] \bmod 1$.

If $q_{N(k)+2} \in A \cup B$, then in Lemma 2 take $x = q_{N(k)+2}$, $y = q_{N(k)+1}$, and $Q = I(k)$. If $q_{N(k)+1} \in A$, let $a = \zeta$ and $b = \zeta + (1/2) - 4\delta$; if $q_{N(k)+1} \in B$, let $a = \varrho$ and $b = \varrho + (1/2) - 4\delta$. If $q_{N(k)+2} \in A$, let $c = \zeta$ and $d = \zeta + (1/2) - 4\delta$; if $q_{N(k)+2} \in B$, let $c = \varrho$ and $d = \varrho + (1/2) - 4\delta$. Since $\delta < 1/48$, it follows that

$$\eta = q_{N(k)+2}/q_{N(k)+1} \geqslant 3/2 > (1+8\delta)/(1-8\delta) = (1-d+c)/(b-a).$$

Since $q_{N(k)+1}/q_{N(k)} \geqslant \mu$, it follows that

$$|Q| = |I(k)| \geqslant 5/(2q_{N(k)+1}) > (1+b-a)/q_{N(k)+1}.$$

Lemma 2 now yields a closed interval $I(k+1) \subset I(k)$ satisfying $|I(k+1)| > \delta/q_{N(k)+2}$ such that for all $t \in I(k+1)$, $tq_{N(k)+1} \in [a, b] \bmod 1$ and $tq_{N(k)+2} \in [c, d] \bmod 1$. Hence the induction step for case 1 is complete.

Case 2. $q_{N(k)+2}/q_{N(k)+1} \geqslant \mu$. Define $N(k+1) = N(k)+1$ and note that

$$q_{N(k+1)+1}/q_{N(k+1)} = q_{N(k)+2}/q_{N(k)+1} \geqslant \mu.$$

Also, since $q_{N(k)+1}/q_{N(k)} \geqslant \mu \geqslant 5/(2\delta)$, we have $|I(k)| \geqslant \delta/q_{N(k)} \geqslant 5/(2q_{N(k)+1})$. Suppose $q_{N(k)+1} \in C \cup C'$. Set $\tau = \theta$ if $q_{N(k)+1} \in C$; set $\tau = \theta'$ if $q_{N(k)+1} \in C'$. Since $|I(k)| \geqslant 5/(2q_{N(k)+1})$, $I(k)$ contains at least two consecutive points of the form $(l+\tau)/q_{N(k)+1}$, where $l \in Z$. This indeed implies that $I(k)$ contains a closed interval $I(k+1)$ satisfying $|I(k+1)| = \delta/q_{N(k)+1}$ such that $tq_{N(k)+1} \in [\tau-\delta, \tau+\delta] \bmod 1$ for all $t \in I(k+1)$. Now suppose $q_{N(k)+1} \in A \cup B$. Set $\sigma = \zeta$ if $q_{N(k)+1} \in A$; set $\sigma = \varrho$ if $q_{N(k)+1} \in B$. Since $|I(k)| \geqslant 5/(2q_{N(k)+1})$, $I(k)$ contains a closed interval $[(l+\sigma)/q_{N(k)+1}, (l+\sigma+(1/2)-4\delta)/q_{N(k)+1}]$, where $l \in Z$. Thus $I(k)$ contains a closed interval $I(k+1)$ satisfying

$$|I(k+1)| = [(1/2)-4\delta]/q_{N(k)+1} > \delta/q_{N(k)+1}$$

such that for all $t \in I(k+1)$, $tq_{N(k)+1} \in [\sigma, \sigma+(1/2)-4\delta] \bmod 1$. This completes the induction step for case 2.

A nested interval argument very similar to that in the proof of Lemma 3 establishes the following result.

LEMMA 4. *Suppose* $\lambda > 1$ *and define*

$$\omega = \min\{m \in Z^+ \cup \{0\}:\ m \geqslant (2-\lambda)/[2(\lambda-1)]\}.$$

Suppose that $0 < \delta < 1/24$ *is small enough so that* $(1+12\delta)/(1-12\delta) < \lambda$, *and that* α *is large enough so that*

$$\alpha \geqslant \max\{(2\omega+3)/2,\ 4(2\omega+3)/[\lambda(1-12\delta)-(1+12\delta)]\}.$$

Let $A, B, C \subset R^+$ *be disjoint, countable sets such that* $A \cup B \cup C = \{q_j\}$ *has the following properties*:

(1) $q_{j+1}/q_j \geqslant \lambda$;

(2) $q_{j+2}/q_j \geqslant \alpha^2$;
(3) *if* $q_j \in C$, *then* $q_{j+1}/q_j \geqslant \alpha$;
(4) *if* $q_j \in C$ *and* $q_j/q_{j-1} < \alpha$, *then* $q_{j-1} \in B$.

Let Q *be any closed interval satisfying* $|Q| \geqslant (2\omega+3)/2q_1$ *and let* $\{r_j\}$ *be any real sequence. Then there is a* $t \in Q$ *such that*

$$tq_j \in [(-1/4)+3\delta, (1/4)-3\delta] \bmod 1 \quad \textit{for all } q_j \in A,$$

$$tq_j \in [1/4, 3/4] \bmod 1 \quad \textit{for all } q_j \in B,$$

$$tq_j \in [r_j-\delta, r_j+\delta] \bmod 1 \quad \textit{for all } q_j \in C.$$

Suppose that Λ is a lacunary set with lacunary ratio $q > 2$. To show that any blocked set E formed from Λ is an I_0-set, Lemmas 3 and 4 will be applied with the following choices of parameters ν, λ, δ, μ, ω, and α:

$$\nu = \min\{1/2, q-2\} > 0;$$

$$\lambda = \min\{2, q/2\} > 1;$$

Choose $0 < \delta < 1/48$ small enough so that

$$16\delta/(1+8\delta) < \nu \quad \text{and} \quad (1+12\delta)/(1-12\delta) < \lambda;$$

$$\mu = 5/(2\delta);$$

$$\omega = \min\{m \in Z^+ \cup \{0\}: m \geqslant (2-\lambda)/[2(\lambda-1)]\};$$

$$\alpha = \max\{(2\omega+3)/2\delta, 4(2\omega+3)/[\lambda(1-12\delta)+(1+12\delta)]\}.$$

In addition, the proof that E is I_0 will depend on a partition of E based on the parameters ε, r, γ, and s which follow:

$$\varepsilon = \min\{\alpha^{-2}, \mu^{-2}\} > 0;$$

Choose $r \in Z^+$ large enough so that $q^r > \max\{\varepsilon^{-2}, \mu^4, \alpha^8\}$;

$$\gamma = \min\{q/2, (1+\varepsilon)/(1+q^{-1}\varepsilon)\} > 1.$$

Choose $s \in Z^+$ large enough so that $\gamma^s \geqslant \alpha^2$.
Write $E = M \cup N$ where

$$M = \{x \in E: x = k+l \text{ where } k \in K, l \in \Lambda(k), \text{ and } l/k \geqslant \varepsilon\},$$

$$N = \{x \in E: x = k+l \text{ where } k \in K, l \in \Lambda(k), \text{ and } l/k < \varepsilon\}.$$

Proposition 3 follows from the fact that M is a lacunary set. The proofs of lacunarity and the two preparatory lemmas are straightforward and are omitted.

LEMMA 5. *Suppose* $\{q_j\}$ *is a lacunary set where* $q_{j+1}/q_j \geqslant q > 2$ *for all* j. *Let* $j > l \geqslant m > 0$. *Then* $q_j/(q_l+q_m) \geqslant q/2 > 1$.

LEMMA 6. *Suppose* $\{q_j\}$ *is a lacunary set with* $q_{j+1}/q_j \geqslant q > 2$ *for all* j. *Suppose* $j \geqslant k$ *and* $l \geqslant m$.

(i) *If* $q_j+q_k > q_l+q_m$, *then* $l \leqslant j$.

(ii) *If* $1 < (q_j+q_k)/(q_l+q_m) \leqslant q/2$ *then* $j = l$ *and* $m < k$.

(iii) (Uniqueness) *If* $q_j+q_k = q_l+q_m$, *then* $j = l$ *and* $k = m$.

LEMMA 7. *M is a lacunary set with lacunary ratio greater than or equal to* $\gamma = \min\{q/2, (1+\varepsilon)/(1+q^{-1}\varepsilon)\}$.

We now proceed with the partition lemma (Lemma 9), a major tool in the proofs of Propositions 4 and 5. Let K' denote the set of translators $k \in K$ which appear in the expression of some $y \in N$ as $y = k+l$ where $l \in \Lambda(k)$. K' is a lacunary set with lacunary ratio at least $q > 2$. Reindexing if necessary, write $K' = \{k_j\}$ where $k_{j+1}/k_j \geqslant q$. Recall that $\varepsilon > 0$ and $r \in Z^+$ have been specified previously and that $q^r > \varepsilon^{-2}$. For each $1 \leqslant i \leqslant r$ define $K(i) = \{k_{jr+i}\colon j \in Z^+ \cup \{0\}\}$. Each $K(i)$ is a lacunary set with lacunary ratio at least q^r and $K' = \bigcup K(i)$, $1 \leqslant i \leqslant r$.

For each $1 \leqslant i \leqslant r$ let $L(i)$ denote the set of $l \in \Lambda$ which appear in the expression of an $y \in N$ as $y = k+l$ where $k \in K(i)$. Note that for a fixed i every $l \in L(i)$ is associated with somer translator in $K(i)$ and that $L(i)$ is a lacunary set with lacunary ratio at least q. Write $L(i) = \{l_j\}$ where $l_{l+1}/l_j \geqslant q$. For each $1 \leqslant j \leqslant r$ define $L(i,j) = \{l_{kr+j}\colon k \in Z^+ \cup \{0\}\}$. Each $L(i,j)$ is a lacunary set with lacunary ratio at least q^r and $L(i) = \bigcup L(i,j)$, $1 \leqslant j \leqslant r$, for each i, $1 \leqslant i \leqslant r$. The next lemma is an immedate consequence of the lacunarity of each $L(i,j)$ and the inequality $q^r > \varepsilon^{-2}$.

LEMMA 8. *Let i and j be integers satisfying $1 \leqslant i, j \leqslant r$. For each $k \in K(i)$ there is at most one $l \in L(i,j)$ which satisfies $\varepsilon k \leqslant l \leqslant q^{r/2}k$.*

LEMMA 9. *Fix integers $1 \leqslant i, j \leqslant r$. There exists a partition of $K(i)$ into disjoint sets $K(i,j,1)$ and $K(i,j,-1)$ such that for every $l \in L(i,j)$, if l is associated with a translator in $K(i,j,\sigma)$, $\sigma = \pm 1$, then*

$$l \notin \bigcup_{k \in K(i,j,\sigma)} [\varepsilon k, q^{r/2}k).$$

Proof. For convenience in notation we reindex so that $K(i) = \{k_n\}$ where $k_{n+1}/k_n \geqslant q^r$. The desired partition of $K(i)$ will be based on an appropriate partition of the indices $n \in Z^+$. Define the function f on Z^+ in the following manner. If there is no $l \in L(i,j)$ satisfying $\varepsilon \leqslant l/k_n < q^{r/2}$, define $f(n) = n$. If there is an $l \in L(i,j)$ satisfying $\varepsilon \leqslant l/k_n < q^{r/2}$, then, by Lemma 8, l is unique. Also, l is associated with precisely one $k_m \in K(i)$; let $f(n) = m$. Note that in this case, since $k_m+l \in N$, we have $l/k_m < \varepsilon$. Therefore, taking into account $\varepsilon \leqslant l/k_n$, we have $k_n < k_m$. That is, $f(n) > n$ when there is an $l \in L(i,j)$ satisfying $\varepsilon \leqslant l/k_n < q^{r/2}$. Note that in any case, $f(n) \geqslant n$ for all n.

We will now show that there is a partition of Z^+ into two disjoint sets $S(1)$, $S(-1)$ such that if $n \in S(\sigma)$, $\sigma = \pm 1$, and $f(n) > n$, then $f(n) \in S(-\sigma)$. Define the relation R on Z^+ by nRn' if and only if $f^s(n) = f^t(n')$ for some $s, t \in Z^+$, where the powers denote composition. It is easy to check that R is an equivalence relation, and hence Z^+ is the union of disjoint R equivalence classes $\{C_m\}$. For each m we define $S(\sigma) \cap C_m$ in the following way. Let n_m

be the least integer in C_m. Given $n \in C_m$, let $p = p(n)$ dénote the smallest nonnegative integer for which there exists a nonnegative integer r such that $f^p(n_m) = f^r(n)$. (Here we are adopting the convention that $f^0(i) = i$ for all i.) With p defined, let q denote the smallest nonnegative integer such that $f^p(n_m) = f^q(n)$. Put $n \in S((-1)^{p-q}) \cap C_m$. Since p and q are unique, it follows that $p-q$ is well defined and hence $S(-1) \cap C_m$ and $S(1) \cap C_m$ are disjoint. Moreover, it is easily verified that if $f(n) > n$ and $n \in S(\sigma) \cap C_m$, then $f(n) \in S(-\sigma) \cap C_m$.

It follows that the sets $S(\sigma) = \bigcup_m (S(\sigma) \cap C_m)$, $\sigma = \pm 1$, have the claimed properties. Moreover, these properties immediately imply that the sets $K(i, j, \sigma) = \{k_n \in K(i)\colon n \in S(\sigma)\}$, $\sigma = \pm 1$, satisfy the conclusion of this lemma.

The separation of M and N (Proposition 4) depends on appropriate dissections of these two sets. Let i, j, and σ be integers satisfying $1 \leqslant i, j \leqslant r$ and $\sigma = \pm 1$. In view of the partition lemma there is an induced partition of $L(i, j)$ into disjoint sets $L(i, j, 1)$ and $L(i, j, -1)$ such that $l \in L(i, j, \sigma)$ if and only if l is associated with a translator in $K(i, j, \sigma)$. It follows from the partition lemma that $K(i, j, \sigma) \cap L(i, j, \sigma) = \emptyset$ and that $K(i, j, \sigma) \cup L(i, j, \sigma)$ is a lacunary set with lacunary ratio at least ε^{-1}. Define

$$N(i) = \{k+l \in N\colon k \in K(i) \text{ and } l \in \Lambda(k)\},$$

$$N(i, j, \sigma) = \{k+l \in N(i)\colon k \in K(i, j, \sigma) \text{ and } l \in L(i, j, \sigma)\}.$$

Since $N = \bigcup N(i)$, $1 \leqslant i \leqslant r$, and each $N(i) = \bigcup N(i, j, \sigma)$, $1 \leqslant j \leqslant r$, $\sigma = \pm 1$, it follows that N is the union of the $2r^2$ sets $N(i, j, \sigma)$ where $1 \leqslant i, j \leqslant r$ and $\sigma = \pm 1$. By Lemma 7, M is lacunary with lacunary ratio $\geqslant \gamma$. Write $M = \{x_j\}$ where $x_{j+1}/x_j \geqslant \gamma$. Recall that $s \in Z^+$ has been chosen earlier. For each $1 \leqslant m \leqslant s$, define $M(m) = \{x_{js+m}\colon j \in Z^+ \cup \{0\}\}$. Each $M(m)$ is a lacunary set with lacunary ratio at least γ^s and $M = \bigcup M(m)$, $1 \leqslant m \leqslant s$. By Proposition 6 to separate M and N it is enough to separate $N(i, j, \sigma)$ and $M(m)$ for each choice of i, j, m, and σ. For the sake of brevity, set $K^* = K(i, j, \sigma)$, $L^* = L(i, j, \sigma)$, and $N^* = N(i, j, \sigma)$. Also define

$$M(m, 0, K^*) = \{x \in M(m)\colon k < x < 2k \text{ for some } k \in K^*\},$$

$$M(m, 0, L^*) = \{x \in M(m)\colon l < x < 2l \text{ for some } l \in L^*\},$$

$$M(m, 1) = \{x \in M(m)\colon 2k \leqslant x < q^{r/4} k \text{ for some } k \in K^*\},$$

$$M(m, 2) = M(m) \backslash [M(m, 1) \cup M(m, 0, K^*) \cup M(m, 0, L^*)].$$

To separate $M(m)$ and N^* it therefore suffices to separate N^* and each one of the four subsets in the dissection of $M(m)$. Lemma 10, which follows easily from Lemma 5, implies that $M \cap K^* = M \cap L^* = \emptyset$, hence each of the four pieces of $M(m)$ is disjoint from $K^* \cup L^*$.

LEMMA 10. *Suppose $\Lambda = \{q_k\}$ is a lacunary set with $q_{k+1}/q_k \geqslant q > 2$ for $k \in Z^+$. Then $q_i \neq q_j + q_m$; and if $q_i > q_j + q_m$, then $q_i/(q_j + q_m) \geqslant q/2$.*

The proof that N^* is separated from $M(m, 1)$ and $M(m, 2)$ is straightforward and is contained in the following two lemmas.

LEMMA 11. *There is a $t \in R$ which separates N^* and $M(m, 1)$.*

Proof. Apply Lemma 4 with the previously specified α and δ and with $\lambda = 2$, $A = L^*$, $B = K^*$, $C = M(m, 1)$, and $r_j = 0$ for all j. Since each $y \in N^*$ is of the form $y = k+l$ where $k \in B$ and $l \in A$, Lemma 4 yields a $t \in R$ such that $ty \in [3\delta, 1-3\delta] \bmod 1$ for all $y \in N^*$ and $tx \in [-\delta, \delta] \bmod 1$ for all $x \in M(m, 1)$. Consequently $\|tx - ty\| \geqslant 2\delta$ for all $x \in M(m, 1)$ and all $y \in N^*$.

LEMMA 12. *There is a $t \in R$ which separates N^* and $M(m, 2)$.*

Proof. Apply Lemma 4 with $A = L^*$, $B = M(m, 2)$, $C = K^*$, and $r_j = 0$ for all j, and with λ, δ, and α as previously specified.

The separation of N^* from $M(m, 0, K^*)$ and $M(m, 0, L^*)$ depends upon further dissections of the latter two sets. These dissections require the following simple result whose proof is omitted.

LEMMA 13. *Let $x \in M$ and suppose that $q_i < x < 2q_i$ for some $q_i \in \Lambda = \{q_k\}$. Then $x = q_i + q_m$ where $q_m \in \Lambda$ and $q_m \leqslant q^{-1} q_i$.*

If $x \in M(m, 0, K^*)$ and $k \in K^*$ is the translator such that $k < x < 2k$, then by Lemma 13, $x = k+z$ where $z \in \Lambda$ and $z \leqslant q^{-1} k$. This induces the following dissection of $M(m, 0, K^*)$. Set $\Lambda^* = \Lambda \setminus (K^* \cup L^*)$ and define

$$M(m, 0, K^*, L^*) = \{k+z \in M(m, 0, K^*):\ k \in K^*, z \in L^*, z \leqslant q^{-1} k\},$$
$$M(m, 0, K^*, K^*) = \{k+z \in M(m, 0, K^*):\ k \in K^*, z \in K^*, z \leqslant q^{-1} k\},$$
$$M(m, 0, K^*, \Lambda^*) = \{k+z \in M(m, 0, K^*):\ k \in K^*, z \in \Lambda^*, z \leqslant q^{-1} k\}.$$

In a similar manner $M(m, 0, L^*)$ can be dissected into

$$M(m, 0, L^*, K^*) = \{l+z \in M(m, 0, L^*):\ l \in L^*, z \in K^*, z \leqslant q^{-1} l\},$$
$$M(m, 0, L^*, L^*) = \{l+z \in M(m, 0, L^*):\ l \in L^*, z \in L^*, z \leqslant q^{-1} l\},$$
$$M(m, 0, L^*, \Lambda^*) = \{l+z \in M(m, 0, L^*):\ l \in L^*, z \in \Lambda^*, z \leqslant q^{-1} l\}.$$

It follows easily from the partition lemma and the definition of blocked sets that $M(m, 0, L^*, K^*) = \emptyset$. Hence to separate N^* from $M(m, 0, K^*)$ and $M(m, 0, L^*)$, it suffices to separate N^* from $M(m, 0, K^*, K^*)$ and from $M(m, 0, X, Y)$ for each $X \in \{K^*, L^*\}$ and $Y \in \{L^*, \Lambda^*\}$.

LEMMA 14. *There is a $t \in R$ which separates N^* and $M(m, 0, K^*, K^*)$.*

Proof. Apply Lemma 3 with ν, δ, and μ as previously specified and with $A = B = \emptyset$, $C = K^*$, $C' = L^*$, $\theta = 0$, and $\theta' = 1/2$. Since each $x \in M(m, 0, K^*, K^*)$ is of the form $x = k+z$ where $k, z \in C$ and each $y \in N^*$ is of the form $y = k'+l$ where $k' \in C$ and $l \in C'$, Lemma 3 yields a $t \in R$ such that $tx \in [-2\delta, 2\delta] \bmod 1$ for all $x \in M(m, 0, K^*, K^*)$ and $ty \in [(1/2)-2\delta,$

$(1/2)+2\delta]$ mod 1 for all $y\in N^*$. Therefore $\|tx-ty\|\geqslant(1/2)-4\delta>0$ for all $x\in M(m, 0, K^*, K^*)$ and all $y\in N^*$.

By interchanging K^* and L^* in the statement and proof of Lemma 14 we have the following result.

LEMMA 15. *There is a $t\in R$ which separates N^* and $M(m, 0, L^*, L^*)$.*

Next we will prove that N^* is separated from $M(m, 0, L^*, \Lambda^*)$ (Lemma 18). Recall $\Lambda^* = \Lambda\backslash(K^*\cup L^*) = \{z_j\}$ is a lacunary set where $z_{j+1}/z_j \geqslant q$ for all j. For each integer $1\leqslant p\leqslant r$, define

$$\begin{aligned}
\Lambda^*(p) &= \{z_{jr+p}\colon j\in Z^+\cup\{0\}\},\\
M^*(L^*, p) &= \{l+z\in M(m, 0, L^*, \Lambda^*)\colon l\in L^*, z\in\Lambda^*(p)\},\\
M^*(L^*, p, 1) &= \{l+z\in M^*(L^*, p)\colon l\in L^*, z\in\Lambda^*(p),\\
&\qquad l' < z < \varepsilon^{-1/2}l' \text{ for some } l'\in L^*\},\\
M^*(L^*, p, 2) &= M^*(L^*, p)\backslash M^*(L^*, p, 1).
\end{aligned}$$

LEMMA 16. *There is a $t\in R$ which separates N^* and $M^*(L^*, p, 1)$.*

Proof. Let $\Lambda^*(p, 1) = \{z\in\Lambda^*(p)\colon l' < z < \varepsilon^{-1/2}l'$ for some $l'\in L^*\}$. Apply Lemma 3 with ν, δ, and μ as previously specified and with $A = L^*$, $B = \emptyset$, $C = K^*$, $C' = \Lambda^*(p, 1)$, $\zeta = (-1/4)+2\delta$, $\theta = 0$, and $\theta' = 1/2$.

LEMMA 17. *There is a $t\in R$ which separates N^* and $M^*(L^*, p, 2)$.*

Proof. Apply Lemma 4 with the previously specified α and δ and with $\lambda = 2$, $A = K^*$, $B = \Lambda^*(p)\backslash\Lambda^*(p, 1)$, $C = L^*$, and $r_j = 0$ for all j.

Since $M(m, 0, L^*, \Lambda^*) = \bigcup M^*(L^*, p)$, $1\leqslant p\leqslant r$, and each $M^*(L^*, p) = M^*(L^*, p, 1)\cup M^*(L^*, p, 2)$, the next result is a corollary to Lemmas 16 and 17 and Proposition 6.

LEMMA 18. *N^* and $M(m, 0, L^*, \Lambda^*)$ are separated.*

LEMMA 19. *N^* and $M(m, 0, K^*, \Lambda^*)$ are separated.*

Proof. Analogous to that of Lemma 18.

Next we focus on the final step in the separation of N^* from $M(m)$: the proof that N^* and $M(m, 0, K^*, L^*)$ are separated (Lemma 21). As defined, each $x\in M(m, 0, K^*, L^*)$ is of the form $x = k+l$ where $k\in K^*$, $l\in L^*$, and $l\leqslant q^{-1}k$. We assert that l is the translator of x. Otherwise, k is the translator. Then $\varepsilon\leqslant l/k$ by virtue of the definition of M. But then $\varepsilon < l/k \leqslant q^{-1}$, which contradicts the partition lemma. Let T denote the set of translators of the elements in $M(m, 0, K^*, L^*)$, and let U denote the set of elements $z\in\Lambda$ such that z is associated with some translator $y\in T$ and $y+z\in M(m, 0, K^*, L^*)$. We have just shown that $T\subset L^*$ and $U\subset K^*$. Note that this implies $T \cap K^* = \emptyset$. Let $k_1 < k_2 < k_3 < \dots$ be an enumeration of $T \cup K^*$. Write $A_0 = M(m, 0, K^*, L^*)$ and $A_1 = N^*$. Then we have $A_0 \cup A_1 = \bigcup (k_n + W_n)$, $n \in Z^+$, where $W_n \subset \Lambda(k_n)$ is uniquely determined by this expression and where T, U, k_n, W_n satisfy:

(1) $T \subset L^*$ and $U \subset K^*$;

(2) If $k_n \in T$, then $W_n \subset U$ and $k_n + W_n \subset A_0$;

(3) If $k_n \in K^*$, then $W_n \subset L^*$ and $k_n + W_n \subset A_1$;

(4) If $n, m \in Z^+$ and $n \neq m$, then $W_n \cap W_m = \emptyset$.

The separation of A_0 and A_1 depends on the following partition result. Its proof, which is derived from properties (1) through (4) and the definition of K^* and L^*, is omitted.

Lemma 20. *There exists a partition of $K^* \cup L^* = \bigcup (\{k_n\} \cup W_n)$, $n \in Z^+$, into four disjoint sets F_0, F_1, F_2, F_3 such that for each $n \in Z^+$ if $k_n + W_n \subset A_l$, then $k_n \in F_s$ and $W_n \subset F_t$ where either $s+t \equiv 2l \pmod 4$ or $s+t \equiv 2l + 1 \pmod 4$.*

Lemma 21. *There is a $t \in R$ which separates N^* and $M(m, 0, K^*, L^*)$.*

Proof. Apply Lemma 4 with λ, α, and δ as previously specified and with $A = B = \emptyset$ and $C = K^* \cup L^*$. Let $C = F_0 \cup F_1 \cup F_2 \cup F_3$ be the partition of C that Lemma 20 yields. Since C is a lacunary set with lacunary ratio at least ε^{-1}, we may write $C = \{q_j\}$ where $q_{j+1}/q_j \geqslant \varepsilon^{-1}$. Define a sequence $\{r_j\}$ taking the values 0, 1/4, 1/2, 3/4 by setting $r_j = i/4$ if and only if $q_j \in F_i$. Lemma 4 yields a $t \in R$ such that for all $j \in Z^+$, $tq_j \in [(i/4) - \delta, (i/4) + \delta] \bmod 1$ where $q_j \in F_i$. Therefore Lemma 20 implies that $tx \in [-2\delta, 2\delta] \cup [(1/4) - 2\delta, (1/4) + 2\delta] \bmod 1$ for all $x \in A_0 = M(m, 0, K^*, L^*)$ and $ty \in [(1/2) - 2\delta, (1/2) + 2\delta] \cup [(3/4) - 2\delta, (3/4) + 2\delta] \bmod 1$ for all $y \in A_1 = N^*$. Consequently $\|tx - ty\| \geqslant (1/4) - 4\delta > 0$ for all $x \in M(m, 0, K^*, L^*)$ and all $y \in N^*$.

The proof of Proposition 5 is based on the decomposition $N = \bigcup N(i, j, \sigma)$, $1 \leqslant i, j \leqslant r$, $\sigma = \pm 1$. We will show that any two distinct pieces in the dissection of N, $N(i, j, \sigma)$ and $N(m, p, \tau)$, are separated (Lemma 24). Thus by Proposition 6 the proof of Proposition 5 is reduced to being able to separate any two disjoint subsets of a given $N(i, j, \sigma)$. That this separation is possible is the content of the next lemma.

Lemma 22. *Let $1 \leqslant i, j \leqslant r$ and $\sigma = \pm 1$. If $N(i, j, \sigma) = E_0 \cup E_1$ is any partition, then there is a $t \in R$ which separates E_0 and E_1.*

Proof. Each $x \in N(i, j, \sigma)$ is of the form $x = k + l$ where $k \in K(i, j, \sigma)$ and $l \in L(i, j, \sigma)$. Define

$$L_0^* = \{l \in L(i, j, \sigma) \colon k + l \in E_0 \text{ for some } k \in K(i, j, \sigma)\},$$

$$L_1^* = \{l \in L(i, j, \sigma) \colon k + l \in E_1 \text{ for some } k \in K(i, j, \sigma)\}.$$

Apply Lemma 3 with v, δ, and μ as previously specified and with $A = B = \emptyset$, $C = K(i, j, \sigma) \cup L_0^*$, $C' = L_1^*$, $\theta = 0$, and $\theta' = 1/2$.

We now begin the proof that $N(i, j, \sigma)$ and $N(m, p, \tau)$, $(i, j, \sigma) \neq (m, p, \tau)$, are separated. For convenience in notation set $N_0^* = N(i, j, \sigma)$, $K_0^* = K(i, j, \sigma)$, $L_0^* = L(i, j, \sigma)$, $N_1^* = N(m, p, \tau)$, $K_1^* = K(m, p, \tau)$, and $L_1^* = L(m, p, \tau)$. The basic strategy of the proof is to obtain an appropriate dissection of N_0^* and show that N_1^* and each piece of N_0^* are separated. With this end in mind, we make the following definitions.

$$L_0^*(0) = L_0^* \cap (K_1^* \cup L_1^*) = L_0^* \cap K_1^*,$$
$$L_0^*(1) = \{l \in L_0^*:\ x < l < \varepsilon^{-1/2} x \text{ for some } x \in K_1^* \cup L_1^*\},$$
$$L_0^*(2) = L_0^* \setminus (L_0^*(0) \cup L_0^*(1)),$$
$$K_0^*(0) = K_0^* \cap (K_1^* \cup L_1^*),$$
$$K_0^*(1) = \{k \in K_0^*:\ x < k < \varepsilon^{-1/2} x \text{ for some } x \in K_1^* \cup L_1^*\},$$
$$K_0^*(2) = K_0^* \setminus (K_0^*(0) \cup K_0^*(1)).$$

For each ordered pair of integers (u, v) where $0 \leqslant u, v \leqslant 2$, define

$$D(u, v) = \{k + l \in N_0^*:\ k \in K_0^*(u),\ l \in L_0^*(v)\}.$$

It is clear from the definitions that $N_0^* = \bigcup D(u, v)$, $0 \leqslant u, v \leqslant 2$.

LEMMA 23. *N_1^* and $D(u, v)$ are separated for all $0 \leqslant u, v \leqslant 2$.*

Proof. There are six one-dimensional vectors which separate N_1^* from $D(0, 1)$, $D(1, 0)$, $D(1, 1)$, $D(0, 2)$, $D(2, 0)$, and $D(2, 2)$. They are obtained by routine applications of Lemma 3, and the details are omitted.

Next, there is a $t \in R^2$ which separates N_1^* and $D(0, 0)$. Note that $D(0, 0)$ is the union of the two disjoint sets

$$D(0, 0, K_1^*) = \{k + l \in D(0, 0):\ k \in K_0^* \cap K_1^*,\ l \in L_0^* \cap K_1^*\},$$
$$D(0, 0, L_1^*) = \{k + l \in D(0, 0):\ k \in K_0^* \cap L_1^*,\ l \in L_0^* \cap K_1^*\}.$$

To see that N_1^* is separated from $D(0, 0, K_1^*)$, apply Lemma 3 with v, δ, and μ as previously specified and with $A = B = \emptyset$, $C = K_1^*$, $C' = L_1^*$, $\theta = 0$, and $\theta' = 1/2$. To see that N_1^* is separated from $D(0, 0, L_1^*)$, first observe that each element of N_1^* is of the form $k + l$ where $k \in K_1^*$ and $l \in L_1^*$, while each element of $D(0, 0, L_1^*)$ is of the form $y + z$ where $y \in K_0^* \cap L_1^*$ and $z \in L_0^* \cap K_1^*$. Let T denote the set of translators of the elements in $D(0, 0, L_1^*)$, and let U denote the set of elements $z \in \Lambda$ such that z is associated with some translator $y \in T$ and $y + z \in D(0, 0, L_1^*)$. Then we have $T \subset L_1^*$ and $U \subset K_1^*$. Let $k_1 < k_2 < \ldots$ be an ordering of $T \cup K_1^*$ and write $A_0 = D(0, 0, L_1^*)$, $A_1 = N_1^*$. Then $A_0 \cup A_1 = \bigcup (k_n + W_n)$, $n \in Z^+$, and properties (1) through (4) in the paragraph preceding Lemma 20 are satisfied. Consequently, Lemma 20 holds for $\bigcup (\{k_n\} \cup W_n)$, $n \in Z^+$, above. Likewise the proof of Lemma 21 with

$D(0, 0, L_1^*)$ in place of $M(m, 0, K^*, L^*)$ and N_1^* in place of N^* shows that there is a $t \in R$ which separates N_1^* and $D(0, 0, L_1^*)$.

Third, there is a $t \in R^2$ which separates N_1^* and $D(1, 2)$. To each $y \in K_0^*(1)$ there corresponds some $w \in K_1^* \cup L_1^*$ such that $w < y < \varepsilon^{-1/2} w$. Define

$$K_0^*(1, K_1^*) = \{y \in K_0^*(1): w < y < \varepsilon^{-1/2} w \text{ for some } w \in K_1^*\},$$

$$K_0^*(1, L_1^*) = \{y \in K_0^*(1): v < y < \varepsilon^{-1/2} v \text{ for some } v \in L_1^*\}.$$

Note that $D(1, 2) = D(1, K_1^*, 2) \cup D(1, L_1^*, 2)$ where

$$D(1, K_1^*, 2) = \{y+z \in D(1, 2): y \in K_0^*(1, K_1^*), z \in L_2^*(2)\},$$

$$D(1, L_1^*, 2) = \{y+z \in D(1, 2): y \in K_0^*(1, L_1^*), z \in L_0^*(2)\}.$$

To see that N_1^* is separated from $D(1, K_1^*, 2)$, apply Lemma 3 with v, δ, and μ as previously specified and with $A = K_1^*$, $B = L_0^*(2)$, $C = L_1^*$, $C' = K_0^*(1, K_1^*)$, $\zeta = (-1/4)+2\delta$, $\varrho = (1/4)+2\delta$, and $\theta = \theta' = 0$. Likewise, to see that N_1^* is separated from $D(1, L_1^*, 2)$, apply Lemma 3 with $A = L_1^*$, $B = L_0^*(2)$, $C = K_1^*$, $C' = K_0^*(1, L_1^*)$, and v, δ, μ, ζ, ϱ, θ, and θ' as above.

Finally, an argument completely analogous to that of the previous paragraph establishes that there is a $t \in R^2$ which separates N_1^* and $D(2, 1)$. This finishes the proof of Lemma 23.

Since $N_0^* = \bigcup D(u, v)$, $0 \leqslant u, v \leqslant 2$, the next result is a consequence of Proposition 6 and Lemma 23.

LEMMA 24. *If* $(i, j, \sigma) \neq (m, p, \tau)$, *then* $N_0^* = N(i, j, \sigma)$ *and* $N_1^* = N(m, p, \tau)$ *are separated.*

Lemma 24 concludes the proof of Proposition 5 and hence Theorem 1. Suitable technical modifications in the proof of Theorem 1 yield a similar result for restricted blocked sets.

THEOREM 2. *Let* $\Lambda = \{q_j\}$ *be a lacunary set with lacunary ratio* $q > (1+\sqrt{5})/2$. *In addition, suppose there exists a* $v > 0$ *such that* $|2-(q_{j+1}/q_j)| \geqslant v$ *for all* j. *Then any restricted blocked set formed from* Λ *is an* I_0*-set.*

The requirement in Theorem 2 that consecutive elements in Λ have ratio bounded away from 2 is essential to our methods (cf. condition (1) of Lemma 3) and perhaps is a true singularity in the theory.

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