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ON THE EXISTENCE OF COMMUTATIVE BANACH–LIE GROUPS WHICH DO NOT ADMIT CONTINUOUS UNITARY REPRESENTATIONS

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0. One of the basic tools in the theory of groups is the investigation of their unitary representations. For locally compact groups this is particularly fruitful owing to the existence of sufficiently many irreducible continuous unitary representations (the theorem of Gelfand–Raikov).

For groups which are not locally compact the situation becomes more complicated. For instance, W. Herer and J. P. R. Christensen gave in [2] an example of a Polish vector space which, when considered as a topological group, admits no non-trivial continuous unitary representations.

In this note we exhibit that the same pathological situation can occur in the case of commutative Lie groups (we use terminology and definitions from Bourbaki [1]). Suitable examples of groups locally isomorphic to various Banach spaces are given in Theorems 4, 5 and 6.

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1. Let H be a complex Hilbert space, and let G be a topological group. A *representation* of G in H is a homomorphism $T\colon G \to \mathrm{GL}(H)$ into the group of automorphisms of H. A representation T is *weakly* (*strongly*) *continuous* if it is continuous in the weak (strong) topology on $\mathrm{GL}(H)$.

A representation T is called *unitary* if all the operators $T(g)$, $g \in G$, are unitary. A one-dimensional unitary representation is called a *character*. For unitary representations the notions of weak and strong continuity coincide.

A representation T is called *faithful* if it is injective, and *trivial* if $T(g) = \mathrm{id}_H$ for all $g \in G$.

A topological group G is called *bounded* if for each open $V \ni 1$ there is an n such that $V^n = G$. If $G = E/K$, where K is a subgroup of a normed space E, then G is bounded iff there is an $r > 0$ such that $E = K + rB$, where B is the unit ball in E.

Let F denote the space of all real measurable functions on (0, 1), with the topology of convergence in measure (as usually we do not distinguish two functions equal almost everywhere). F is a separable Fréchet space with F-norm given by

$$|f| = \int_0^1 \min(1, |f(t)|)\,dt.$$

For a given real normed space E let $F(E)$ denote the Fréchet space of all continuous linear operators $A: E \to F$ with F-norm given by $|A| = \sup\{|Au|: \|u\| \leqslant 1\}$.

2. PROPOSITION 1. *Let K be a subgroup of a real normed space E. If the quotient group E/K admits a non-trivial continuous character, then there exists an $f \in E^*$, $f \neq 0$, such that $f(K) \subset Z$.*

This is a simple fact and we leave it without proof.

PROPOSITION 2. *Let K be a subgroup of a real normed space E. If the quotient group E/K admits a non-trivial strongly continuous unitary representation in a separable Hilbert space, then there exists an operator $A \in F(E)$, $A \neq 0$, such that for each $u \in K$ the function Au assumes only integer values.*

This is an immediate consequence of Theorem 5 from [2].

PROPOSITION 3. *Let G be a bounded commutative topological group satisfying the first axiom of countability, and let T be a weakly continuous representation of G in a Hilbert space H. Then T is equivalent to a unitary representation.*

Proof. Let (g_n) be any sequence converging to zero in G. For each $u \in H$ the sequence $(T(g_n)u)$ converges weakly to u. In particular the set of its terms is bounded. Hence, in view of the Banach–Steinhaus theorem, we have

$$\sup\{\|T(g_n)\|: n \in N\} < \infty.$$

Since $g_n \to 0$ was arbitrary and G has a countable basis of neighbourhoods of zero, there is a neighbourhood V of zero such that

$$L = \sup\{\|T(g)\|: g \in V\} < \infty.$$

From the boundedness of G we have $G \subset V^n$ for a certain n. Then

$$\sup\{\|T(g)\|: g \in G\} \leqslant L^n < \infty.$$

On the commutative group G there is an invariant mean, and it is enough to repeat the standard argument for compact groups.

3. THEOREM 4. *Let E be an infinite dimensional real normed space, for which the space $F(E)$ is separable. Then there exists a discrete subgroup $K \subset E$ such that the quotient group E/K admits no non-trivial weakly continuous representations in Hilbert spaces.*

Proof. Let X be the set $\boldsymbol{Z}+\langle 1/4, 3/4\rangle \subset \boldsymbol{R}$, and let m be the Lebesgue measure on (0, 1). It is a matter of technique to prove the existence of a positive constant c such that for every $0 \neq A \in F(E)$ we have

$$\text{(i)} \qquad W_A + c^{-1}|A|^{-1} B = E,$$

where B is the unit ball in E, and W_A – the set

$$\{u \in E\colon\ m(\{t \in (0, 1)\colon\ Au(t) \in X\}) \geqslant c|A|\}.$$

The separability of $F(E)$ implies the separability of E^*. Therefore E itself is separable, so we can choose a set $\{b_n\}_{n=1}^{\infty}$ dense in E, and another set $\{A_n\}_{n=1}^{\infty}$, $A_n \neq 0$, dense in $F(E)$.

We shall build by induction a sequence $(a_n)_{n=1}^{\infty}$ in E. By (i), there is an $a_1 \in W_{A_1}$ such that

$$1 \leqslant \|a_1\| \leqslant 2c^{-1}|A_1|^{-1}+1.$$

After finding $a_1, \ldots, a_{2n-1}$ choose $a_{2n} \in b_n + B$ such that

$$d(a_{2n}, \operatorname{span}\{a_1, \ldots, a_{2n-1}\}) \geqslant 1.$$

Then, using (i) again, choose $a_{2n+1} \in W_{A_{n+1}}$ such that

$$d(a_{2n+1}, \operatorname{span}\{a_1, \ldots, a_{2n}\}) \geqslant 1,$$

and

$$\|a_{2n+1}\| \leqslant 2c^{-1}|A_{n+1}|^{-1}+1,$$

and so on.

The sequence (a_n) constructed in this way satisfies the following conditions:

$$\text{(ii)} \qquad d(a_n, \operatorname{span}\{a_k\colon k < n\}) \geqslant 1,$$

$$\text{(iii)} \qquad \|a_{2n-1}\| \leqslant 2c^{-1}|A_n|^{-1}+1,$$

$$\text{(iv)} \qquad a_{2n-1} \in W_{A_n},$$

$$\text{(v)} \qquad a_{2n} \in b_n + B$$

for $n = 1, 2, \ldots$ Let K be the group generated by the vectors a_n, $n = 1, 2, \ldots$ From (ii) it follows easily that $\|u-v\| \geqslant 1$ for any different $u, v \in K$; hence K is discrete. Next, for any given $u \in E$ we can find $b_n \in u+B$, and then

$$u \in b_n + B = a_{2n} + (b_n - a_{2n}) + B \subset K + B + B$$

by (v). Hence $E = K+2B$ and E/K is bounded.

Suppose now that E/K admits a non-trivial weakly continuous representation T in a Hilbert space H. According to Proposition 3 we may assume that T is unitary. Then T is a Hilbert sum of cyclic representations, and we may assume that T itself is cyclic. From the separability of E/K it follows now that H is separable. Then, in view of Proposition 2, there is a non-zero $A \in F(E)$ such that $\operatorname{im} Au \subset \boldsymbol{Z}$ for all $u \in K$.

Let x be the largest integer $\leqslant 2c^{-1}|A|^{-1}+2$. Since $\{A_n\}$ is dense in $F(E)$, we can find an n for which $|A_n| \geqslant |A|$ and $|A_n - A| < c|A|/4x$. Then (iii) gives

$$|(A_n - A)\, a_{2n-1}| \leqslant x\,|(A_n - A)\, x^{-1} a_{2n-1}| \leqslant x\,|A_n - A| < c\,|A|/4.$$

On the other hand, from (iv) and the definition of W_{A_n} it follows that there is a subset $S \subset (0, 1)$ such that $m(S) \geqslant c|A|$, and $(A_n a_{2n-1})(S) \subset X$. Then

$$|(A_n - A)\, a_{2n-1}| \geqslant \int_S \min\bigl(1, |A_n a_{2n-1}(t) - A a_{2n-1}(t)|\bigr)\, dt$$

$$\geqslant \int_S dt/4 \geqslant c\,|A|/4,$$

because Aa_{2n-1} assumes integer values a.e. This contradiction completes the proof.

In a similar way, using transfinite induction, we can prove that if an infinite dimensional real normed space E has the property $\operatorname{wght} E = \operatorname{wght} F(E)$ (it is always true that $\operatorname{wght} E \leqslant \operatorname{wght} F(E)$), then there exists a discrete subgroup $K \subset E$ such that the group E/K admits no non-trivial weakly continuous representations in separable Hilbert spaces.

Using the same method, with some simplifications, we can prove the following

THEOREM 5. *Let E be an infinite dimensional real normed space such that* $\operatorname{wght} E = \operatorname{wght} E^*$. *Then there exists a discrete subgroup $K \subset E$ such that the group E/K admits no non-trivial continuous characters.*

It is known that the spaces $F(c_0)$ and $F(l^p)$, $2 < p < \infty$, are separable, while $F(l^p)$, $1 \leqslant p \leqslant 2$, are not. Similarly, the spaces $(c_0)^*$ and $(l^p)^*$, $1 < p < \infty$, are separable, and $(l^1)^*$ is not. However, in this special case there is a more straightforward method of constructing exotic groups than using Theorems 4 and 5.

For each $n \in N$ let e_n be the sequence $(\delta_{nk})_{k=1}^{\infty}$, where δ_{nk} is the Kronecker delta. Let then $(a_n)_{n=1}^{\infty}$ be an arbitrary sequence dense in the real space l^1, such that $a_1 = 0$ and

(vi) $$a_n \in \operatorname{span}\{e_1, \ldots, e_{n-1}\} \quad \text{for} \quad n \geqslant 2.$$

For the group K generated by the vectors $a_n + e_n$, $n = 1, 2, \ldots$, we have the following

THEOREM 6. *K is a discrete subgroup in each of the real Banach spaces c_0 and l^p, $1 \leqslant p < \infty$. The corresponding quotient groups c_0/K and l^p/K are commutative, connected, bounded, separable, infinite dimensional Lie groups. The groups c_0/K and l^p/K, $p > 1$, do not admit any non-trivial continuous characters. The groups c_0/K and l^p/K, $p > 2$, do not admit any non-trivial weakly continuous representations in Hilbert spaces.*

Proof. It follows immediately from (vi) that K is discrete in l^p and in c_0. Therefore c_0/K and l^p/K are infinite dimensional Lie groups. It is obvious that they are commutative, connected and separable. We shall prove that they are bounded.

Given any $u \in l^p$ ($u \in c_0$) we can find an n such that $\|u-a_n\| \leqslant 1$, because $\{a_n\}$ is dense in l^1, and therefore also in c_0 and all l^p. Then

$$\|u-a_n-e_n\| \leqslant \|u-a_n\|+\|e_n\| \leqslant 2.$$

From this we have $K+2B = l^p$ ($K+2B = c_0$), where B is the unit ball, and the boundedness is proved.

Suppose now that the group E/K, where E is c_0 or l^p, $p > 1$, admits a non-trivial continuous character. Then by Proposition 1 there is a non-zero $f \in E^*$ such that $f(K) \subset \boldsymbol{Z}$. For a certain $a \in E$ we have $f(a) \notin \boldsymbol{Z}$. Let (a_{k_n}) be a subsequence of (a_n) converging to a. Then the sequence

$$f(e_{k_n}) = f(a_{k_n}+e_{k_n})-f(a_{k_n})$$

does not converge to zero, which is impossible in l^p, $p > 1$, and in c_0.

Now let us suppose that the group E/K, where E is the space c_0 or l^p, $p > 2$, admits a non-trivial weakly continuous representation in a Hilbert space H. By Proposition 3 and the boundedness of E/K, T may be assumed to be unitary. Since E/K is separable and T is a Hilbert sum of cyclic representations, we may assume that H is separable. Then Proposition 2 gives the existence of an $A \in F(E)$, $A \neq 0$, such that $\operatorname{im} Au \subset \boldsymbol{Z}$ for $u \in K$.

Since $A \neq 0$, there is an $a \in E$ such that the set

$$S = \{t\colon (Aa)(t) \in \boldsymbol{Z}+\langle 2/5, 3/5\rangle\}$$

has a positive measure. Let (a_{k_n}) be a subsequence of (a_n) converging to a. The sequence (Aa_{k_n}) converges in measure to the function Aa. Using the Egorov theorem we may assume that $Aa_{k_n} \to Aa$ uniformly on S. Hence

$$(Aa_{k_n})(t) \in \boldsymbol{Z}+\langle 1/4, 3/4\rangle \quad \text{for} \quad t \in S \text{ and } n \in \boldsymbol{N}.$$

All the functions $A(a_{k_n}+e_{k_n})$ assume integer values only, therefore $(Ae_{k_n})(t) \in \boldsymbol{Z}+\langle 1/4, 3/4\rangle$ for $t \in S$. In particular

(vii) $$|(Ae_{k_n})(t)| \geqslant 1/4 \quad \text{for} \quad t \in S \text{ and } n \in \boldsymbol{N}.$$

Put $f_n = n^{-1/2}Ae_{k_n}$ for $n = 1, 2, \ldots$ Each subsequence of the sequence $(n^{-1/2}e_{k_n})$ converges in E. Therefore, in view of the continuity of A, each subsequence of (f_n) converges in F. Hence $\sum f_n^2 < \infty$ a.e., by the Orlicz theorem (see [3], Lemma). But, in view of (vii), the series $\sum f_n^2$ is divergent on S. This contradiction completes the proof.

It is not difficult to show that the group l^1/K admits sufficiently many continuous characters, and that each of the groups l^p/K, $1 \leqslant p \leqslant 2$, admits a faithful strongly continuous unitary representation in a separable Hilbert space.

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