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AN ERGODIC THEOREM WITHOUT INVARIANT MEASURE

BY

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TO THE MEMORY OF JANUSZ WOŚ

The following is proved:

If τ is a conservative, non-singular automorphism of a probability space $(X, \mathfrak{F}, \mu)$, then to each real measurable function h such that

$$\sup_{n\geq 0}\left|\sum_{j=0}^{n} h(\tau^j x)\frac{d\mu\circ\tau^j}{d\mu}(x)\Big/\sum_{j=0}^{n}\frac{d\mu\circ\tau^j}{d\mu}(x)\right| < \infty \quad \text{on } X$$

there corresponds a real function $\tilde{h}$, measurable with respect to the σ-field of all τ-invariant subsets of X, such that

$$\tilde{h}(x) = \lim_{\substack{m+n\to\infty\\ m,n\geq 0}} \sum_{j=-m}^{n} h(\tau^j x)\frac{d\mu\circ\tau^j}{d\mu}(x)\Big/\sum_{j=-m}^{n}\frac{d\mu\circ\tau^j}{d\mu}(x) \quad \text{on } X.$$

A point transformation τ from X onto itself is called a *non-singular automorphism* of $(X, \mathfrak{F}, \mu)$ if

(i) τ is invertible;

(ii) $A\in\mathfrak{F}$ implies $\tau^{-1}A, \tau A\in\mathfrak{F}$;

(iii) $\mu A = 0$ implies $\mu(\tau^{-1}A) = \mu(\tau A) = 0$.

In this note τ will be assumed to be a non-singular automorphism of $(X, \mathfrak{F}, \mu)$. Thus to each integer j there corresponds the Radon–Nikodym derivative

$$w_j(x) = \frac{d\mu\circ\tau^j}{d\mu}(x).$$

Let us write, for any measurable function h on X,

$$T^j h(x) = h(\tau^j x) w_j(x).$$

Since $w_{i+j}(x) = w_i(x) w_j(\tau^i x)$ on X, it follows that

$$T^{i+j}h(x) = T^i[T^j h](x) \quad \text{on } X.$$

Further, $\|T^j h\|_1 = \|h\|_1$ for all $h\in L_1(\mu)$. Therefore the operator $T = T^1$ may be regarded as a positive invertible isometry of $L_1(\mu)$. (It is known that if $(X, \mathfrak{F}, \mu)$ is a Lebesgue space, then any positive invertible isometry of $L_1(\mu)$ has this form.) τ is called *conservative* if $\tau^{-1}A\subset A$ and $A\in\mathfrak{F}$ imply $\mu(A\setminus\tau^{-1}A) = 0$.

τ is conservative if and only if

$$\sum_{j=0}^{\infty} T^j 1(x) = \sum_{j=0}^{\infty} w_j(x) = \infty \quad \text{on } X$$

(see, e.g., Section 3.1 in [4]). $A \in \mathfrak{F}$ is called *τ-invariant* if

$$\tau^{-1}A = A \ (\mathrm{mod}\,\mu)$$

(i.e., $\mu(A \Delta \tau^{-1}A) = 0$). Let $\mathfrak{I}$ denote the family of all τ-invariant sets; $\mathfrak{I}$ forms a sub-σ-field of $\mathfrak{F}$.

When τ preserves μ (i.e., $\mu(\tau^{-1}A) = \mu A$ for all $A \in \mathfrak{F}$), we have $w_j(x) = 1$ on X and $T^j h(x) = h(\tau^j x)$ for each integer j; if h is a real measurable function on X and $n^{-1} \sum_{j=0}^{n-1} h(\tau^j x)$ is bounded for a.e. $x \in X$, then, by Kesten's ergodic theorem (see p. 211 in [3]), $n^{-1} \sum_{j=0}^{n-1} h(\tau^j x)$ is convergent for a.e. $x \in X$. Kesten proved this result by using some previous results of Tanny ([5], [6]); Woś ([7], [8]) gave a simple proof of this result (see also [1]).

In this note we will use the method of Woś to generalize Kesten's theorem to non-invariant measures.

THEOREM. *Let τ be a conservative, non-singular automorphism of a probability space $(X, \mathfrak{F}, \mu)$ and let h be a real measurable function. Write*

$$\bar{h}(x) = \limsup_{n\to\infty} \sum_{j=0}^{n} h(\tau^j x) w_j(x) \Big/ \sum_{j=0}^{n} w_j(x).$$

Then $\bar{h}$ is measurable with respect to $\mathfrak{I}$, and

$$\bar{h}(x) = \liminf_{n\to\infty} \sum_{j=0}^{n} h(\tau^{-j} x) w_{-j}(x) \Big/ \sum_{j=0}^{n} w_{-j}(x) \tag{1}$$

for a.e. x in the set $\{\bar{h} < \infty\}$.

Proof. Since

$$\bar{h} = \limsup_{n\to\infty} \sum_{j=0}^{n} T^j h \Big/ \sum_{j=0}^{n} T^j 1,$$

we have

$$T\bar{h}(x) = w_1(x)\bar{h}(\tau x) = w_1(x) \limsup_{n\to\infty} \frac{w_1(x) \sum_{j=0}^{n} T^j h(\tau x)}{w_1(x) \sum_{j=0}^{n} T^j 1(\tau x)}$$

$$= w_1(x) \limsup_{n\to\infty} \frac{\sum_{j=1}^{n+1} T^j h(x)}{\sum_{j=1}^{n+1} T^j 1(x)} = w_1(x)\bar{h}(x),$$

where the last equality is due to the fact that

$$\sum_{j=0}^{\infty} T^j 1(x) = \infty \quad \text{on } X.$$

Thus $\tilde{h}(x) = \tilde{h}(\tau x)$ on X, and $\tilde{h}$ is measurable with respect to $\mathfrak{I}$.

To prove (1), let $X_N = \{\tilde{h} < N\}$. Since

$$X_N \in \mathfrak{I} \quad \text{and} \quad \bigcup_{N=1}^{\infty} X_N = \{\tilde{h} < \infty\},$$

it suffices to concentrate our attention on the set X_N. Then, considering $h-N$ instead of h, it may be assumed that

$$\{\tilde{h} < 0\} = X.$$

Under this assumption we get

$$H(x) = \sup_{n \geqslant 0} \sum_{j=0}^{n} T^j h(x) < \infty \quad \text{on } X.$$

Therefore

(2) $$H^+(x) < \infty \quad \text{and} \quad h(x) = -H^-(x) + H^+(x) - T[H^+](x),$$

because

$$H(x) = h(x) + \sup_{n \geqslant 1} \Big[\sum_{j=1}^{n} T^j h(x)\Big]^+ = h(x) + \sup_{n \geqslant 1} \Big[T\Big(\sum_{j=0}^{n-1} T^j h\Big)(x)\Big]^+$$

$$= h(x) + w_1(x) \sup_{n \geqslant 0} \Big[\sum_{j=0}^{n} T^j h(\tau x)\Big]^+ = h(x) + T[H^+](x) < \infty.$$

We now prove that

(3) $$\liminf_{n \to \infty} T^{-n}[H^+] / \sum_{j=0}^{n} T^{-j} 1 = 0 = \liminf_{n \to \infty} T^n [H^+] / \sum_{j=0}^{n} T^j 1$$

on X. To see this, let

$$f_N(x) = \inf_{j \geqslant N} T^{-j}[H^+](x) \quad \text{for } N \geqslant 0.$$

Then

$$0 \leqslant f_N \leqslant f_{N+1} = T^{-1} f_N < \infty \quad \text{on } X.$$

It follows that $0 \leqslant T f_N \leqslant f_N < \infty$ on X. This implies $T f_N = f_N$, since T is a conservative contraction operator on $L_1(\mu)$ (see, e.g., p. 16 in [2]). Consequently, $0 \leqslant f_N = f_{N+1} \leqslant H^+ < \infty$ on X, and the first equality of (3) follows. The second equality follows similarly.

By (2), (3) and the Neveu–Chacon identification theorem of the ratio ergodic limit (see, e.g., Section 3.3 in [4]) we obtain

$$\tilde{h} = -\mathrm{E}\{H^- | \mathfrak{I}\} = \liminf_{n\to\infty} \sum_{j=0}^{n} T^{-j}h / \sum_{j=0}^{n} T^{-j}1,$$

completing the proof.

COROLLARY. *Let τ and h be as in the Theorem. Then the limit*

$$\lim_{\substack{m+n\to\infty \\ m,n\geq 0}} \sum_{j=-m}^{n} h(\tau^j x)w_j(x) / \sum_{j=-m}^{n} w_j(x)$$

exists for a.e. x in the set

$$\{x\colon \sup_{n\geq 0} \Big|\sum_{j=0}^{n} h(\tau^j x)w_j(x) / \sum_{j=0}^{n} w_j(x)\Big| < \infty\}.$$

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