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FINITELY ADDITIVE INVARIANT MEASURES. I

BY

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1. Professor Marczewski urged me several times to write a survey paper on this subject and finally I am trying to satisfy his wish. Let me recall on this occasion that Marczewski often urged his students and colleagues to publish their work. At the meetings of the Polish Mathematical Society he would often ask how and when the speaker intended to publish his results. The activity of young mathematicians in Wrocław was greatly stimulated by his constant friendly interest in their work and his encouragement to overcome this final, but for many not the least, difficulty of writing it down.

Marczewski's efforts bring to my mind other thoughts. One hears critical remarks about the "explosion of science", the increasing difficulty of following the development of mathematics and the irrelevance of many papers. There are mathematicians who advocate restraint in publication and waiting for the maturity of results. But is it not clear that the great number of publications is just a function of the number of working mathematicians, and with its growth, even the best mathematicians can appreciate only a smaller and smaller fraction of what is being published? This does not mean that the quality of mathematical publications has decreased. On the contrary, I believe that with the modern standards of proof and clarity this quality has increased. Also, if weighed against the oceans of nonsense which are published on our planet, almost all mathematical papers seem outstanding. More than once I have been bothered by the unavailability in the literature of various results and proofs which at one time had been circulated in the form of preprints, and then got stuck somewhere before appearing in print. Thus the views mentioned above are wrong and the process of publication is much too slow and often too cumbersome. As to the reader, it remains his task to select what he can learn and enjoy. Clearly, we prefer more degrees of freedom rather than fewer (also in a library).

2. In this paper *measure* will mean finitely additive measure with values in $[0, \infty]$. I want to outline some applications of well-known methods for proving the existence of various measures. Banach constructed a measure m over the Boolean algebra $P(\mathcal{R}^2)$ of *all* subsets of the plane $\mathcal{R}^2$ which extends the ordinary 2-dimensional measure $\lambda^{(2)}$ of Lebesgue and is invariant under isometries of sets. Marczewski realized that Banach's construction can be modified so as to yield another measure μ with the following properties. Like m, μ is defined over $P(\mathcal{R}^2)$ and is invariant under isometries; $\mu(A) = \lambda^{(2)}(A)$ if $\lambda^{(2)}(\text{boundary} A) = 0$ and $\mu(M) = 0$ for all $M \subseteq \mathcal{R}^2$ which are *meager*, i.e., of the first category. He also asked whether a measure with the same properties over the algebra of subsets of $\mathcal{R}^2$ having the property of Baire is possible. This problem has some interesting equivalent formulations (see [13]) but it remains unsolved. Finally, Marczewski also asked [10] if extensions of Hausdorff's α-dimensional measures $\lambda^{(\alpha)}$ similar to Banach's extension m of $\lambda^{(2)}$ are possible. We shall see in the sequel that the answer is affirmative. We shall also provide proofs of the existence of a μ as above with the additional property

$$\mu(s[A]) = a_s\mu(A)$$

for every similarity s of $\mathcal{R}^2$, where a_s is the square of the magnification factor of s. We shall also prove the existence of extensions of $\lambda^{(\alpha)}$ satisfying a similar equation where a_s is the α-th power of the magnification coefficient of s. Finally, refining a theorem of [16], we shall prove the existence of a measure μ over the algebra L_n of all Lebesgue measurable subsets of $\mathcal{R}^n$ such that $\mu(\mathcal{R}^n) = 1$ and $\mu(s[A]) = \mu(A)$ for all $A \in L_n$ and all similarities s of $\mathcal{R}^n$.

For various other results and references related to the material of this paper see [12].

3. Amenable groups. One of the most interesting concepts in this theory is that of an amenable group. A group G is called *amenable* if there exists a universal (i.e., defined over the Boolean algebra $P(G)$ of all subsets of G) finitely additive measure m which is left invariant (i.e., $m(aA) = m(A)$ for all $a \in G$ and $A \subseteq G$) and such that $m(G) = 1$. Følner [5] gave the following purely algebraic characterization of amenability:

(F) *G is amenable iff for every $\varepsilon > 0$ and every finite non-empty set $E \subseteq G$ there exists a finite set $E^* \subseteq G$ such that*

$$\frac{|aE^*\Delta E^*|}{|E^*|} \leqslant \varepsilon \quad \textit{for all } a \in E, \tag{1}$$

where Δ is the symmetric difference of sets and $|X|$ is the number of elements of X.

The sufficiency of this condition is easy to establish by a generalized limit argument. The necessity is much harder (see [6]). It is easy to prove (see [5] and [6]) that the class of amenable groups has the following properties:

3.1. *It is closed under subgroups and homomorphisms.*

3.2. *If all finitely generated subgroups of a group G are amenable, then G is amenable.*

3.3. *The extension of an amenable group by an amenable group is amenable.*

3.4. *All finite groups and all Abelian groups, and hence by* 3.3 *also all solvable groups, are amenable.*

3.5. *An amenable group has a measure m as above which is moreover both left and right invariant (i.e., invariant).*

The next statement is related to the work by Morse [11] and is proved *without using the axiom of choice.*

3.6. *If G is countable and satisfies Følner's condition* (F) *and $\boldsymbol{B}$ is a countable Boolean algebra of subsets of G containing G which is invariant (i.e., if $A \in \boldsymbol{B}$ and $g \in G$, then $gA \in \boldsymbol{B}$ and $Ag \in \boldsymbol{B}$), then there exists an invariant measure m over $\boldsymbol{B}$ with $m(G) = 1$.*

Proof. We let $g_0, g_1, \ldots$ be a sequence containing all elements of G. First we shall prove that for every finite subalgebra $\boldsymbol{B}_0 \subseteq \boldsymbol{B}$ there exists a measure m_0 over $\boldsymbol{B}_0$ such that $m_0(A) = m_0(B)$ whenever $A, B \in \boldsymbol{B}_0$ and there exists a $g \in G$ such that $gA = B$. By compactness it is enough to prove that for every N, $\varepsilon > 0$ there exists a measure m_ε over $\boldsymbol{B}_0$ with $m_\varepsilon(G) = 1$ such that

$$|m_\varepsilon(A) - m_\varepsilon(B)| < \varepsilon \tag{2}$$

whenever there exists an $n \leqslant N$ such that $g_n A = B$. We let

$$E = \{g_0^{-1}, g_1^{-1}, \ldots, g_N^{-1}\}$$

and we put

$$m_\varepsilon(A) = \frac{|A \cap E^*|}{|E^*|} \quad \text{for all } A \in \boldsymbol{B}_0,$$

where E^* satisfies (1). Then (2) follows from (1) and it is obvious that m_ε is a measure over $\boldsymbol{B}_0$ with $m_\varepsilon(G) = 1$.

Thus by compactness of the space $[0, 1]^{\boldsymbol{B}_0}$ we get the required measure m_0 for any finite subalgebra $\boldsymbol{B}_0 \subseteq \boldsymbol{B}$. Now, by compactness of the space $[0, 1]^{\boldsymbol{B}}$, we get a left invariant measure m' over $\boldsymbol{B}$ with $m'(G) = 1$.

Now let $\boldsymbol{B}^*$ be the least Boolean algebra of subsets of G such that $\boldsymbol{B} \subseteq \boldsymbol{B}^*$, $\boldsymbol{B}^*$ is invariant and $S_{A,s,t} \in \boldsymbol{B}^*$ for any $A \in \boldsymbol{B}$ and any rational numbers s, t, where

$$S_{A,s,t} = \{g\colon s \leqslant m'(Ag) < t\}.$$

Since $\boldsymbol{B}$ and G are countable, it is clear that so is $\boldsymbol{B}^*$. Then, by the same procedure as above, we construct a left invariant measure m^* over $\boldsymbol{B}^*$ with $m^*(G) = 1$. Then, since $m' \leqslant 1$, for all $A \in \boldsymbol{B}$ we can define

$$m(A) = \int_G m'(Ag)m^*(dg)$$

or, more precisely,

$$m(A) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{k}{n} m^*(S_{A,k/n,(k+1)/n}).$$

By routine arguments we can prove that m is a measure as required in 3.6.

Finally, we get another well-known fact:

3.7. *If G has a free non-Abelian subgroup, then G is not amenable.*

We are left with the well-known open problem whether there exists a non-amenable group G without free non-Abelian subgroups. In view of the work of Adjan, perhaps the free group in the variety defined by the equation $x^{665} = e$ with two free generators a and b (which is infinite, see [1]) is such a group, and the set $E = \{a, b\}$ violates (1).

There exists another remarkable characterization of amenability which follows from a more general theorem of Tarski ([21], Corollary 14.14). We say that a set $A \subseteq G$ has a *paradoxical decomposition* if there exists a sequence of disjoint sets $A_1, \ldots, A_m, B_1, \ldots, B_n$ in A and of elements $a_1, \ldots, a_m, b_1, \ldots, b_n$ of G such that

$$A \subseteq a_1A_1 \cup \ldots \cup a_mA_m \quad \text{and} \quad A \subseteq b_1B_1 \cup \ldots \cup b_nB_n.$$

Tarski's theorem implies that

3.8. *G is amenable iff G itself does not have paradoxical decompositions.*

Remark. Another of Tarski's results (a generalized version of the Cantor-Schröder-Bernstein theorem) implies that if there exists a paradoxical decomposition of A, then there exists one with the additional property

$$A_1 \cup \ldots \cup A_m \cup B_1 \cup \ldots \cup B_n = a_1A_1 \cup \ldots \cup a_mA_m = b_1B_1 \cup \ldots \cup b_nB_n = A.$$

It is obvious that if G has a paradoxical decomposition, then it is not amenable, and this is the natural way of proving 3.7. Namely, using an idea which goes back to Hausdorff, one proves directly the following result (see [18], Section 10, where a simple proof and some references are given):

3.9. *If G has a free non-Abelian subgroup, then G has paradoxical decompositions (even ones with $m = n = 2$).*

Concerning the non-amenability of SO(3) see Section 7.

Let us mention a property of groups, stronger than amenability, which grew out of some work of von Neumann, Lindenbaum and Tarski. Following Rosenblatt [20], a group G is called ***supramenable*** if for every non-empty set $A \subseteq G$ there exists a left invariant measure m over the algebra of all subsets of G such that $m(A) = 1$. (For relating this definition with the definition given in [20] see Theorem 5.3 in the sequel.) Tarski's result mentioned above ([21], Corollary 14.14) implies also that

3.10. *G is supramenable iff no non-empty set $A \subseteq G$ has paradoxical decompositions.*

It is proved in [20] that

3.11. *Every solvable group without a free non-Abelian subsemigroup is supramenable.*

In particular, all nilpotent groups are supramenable since nilpotent groups are solvable and have no free subsemigroups (see [9]).

3.12. *If a group G has a free non-Abelian subsemigroup, then G is not supramenable.*

Proof. Let $S \subseteq G$ be a free subsemigroup with two free generators a and b. Then S has a paradoxical decomposition. In fact, we have $aS \subseteq S$, $bS \subseteq S$, $aS \cap bS = \emptyset$, $S \subseteq a^{-1}aS$ and $S \subseteq b^{-1}bS$.

For example, the group of transformations $ax+b$ (where $a > 0$) of the real line $\boldsymbol{R}$ and the group of transformations $ax+b$ (where $|a| = 1$) of the complex plane are solvable but have free subsemigroups (of power $2^{\aleph_0}$), and hence are not supramenable. This leads to various constructions related to the proof above (see e.g. [15], Theorem 3 and references therein). The concept of a group of transformations of a space has been generalized by Tarski [21] so as to permit the treatment of isometries of subsets in a metric space such that the isometries do not necessarily extend to the whole space. (See, e.g., [21], Theorem 16.3, for such an extension of 3.11 in the case of Abelian groups.)

All this constitutes of course a rather special chapter of measure theory and group theory, while the main stream deals with countably additive Borel measures (see e.g. [6], [16] and [19]).

4. Extensions of measures. Let $\boldsymbol{B}$ be a Boolean algebra, and $\boldsymbol{R}$ a *subring* of $\boldsymbol{B}$, i.e., a subset of $\boldsymbol{B}$ closed under Boolean join and Boolean subtraction. Let μ be a measure over $\boldsymbol{R}$ with values in $[0, \infty]$.

4.1. EXTENSION THEOREM. *μ can be extended to a measure μ^* over $\boldsymbol{B}$.*

This is a well-known fact. In [8] (see also [14], footnote (13)) Luxemburg proves the equivalence of 4.1 and related statements with the Hahn-Banach theorem without using the axiom of choice. The reader who wants just a straightforward proof of 4.1 is advised to prove first that for every finite Boolean algebra $\boldsymbol{B}_0 \subseteq \boldsymbol{B}$ the measure μ restricted to $\boldsymbol{B}_0 \cap \boldsymbol{R}$ can be extended to $\boldsymbol{B}_0$, and then to use the compactness of the space $[0, \infty]^{\boldsymbol{B}}$ (the Tychonoff theorem), as in the proof of 3.6.

An interesting fact about Theorem 4.1 is that, already for the algebra of all subsets of a countable set, it cannot be proved without using the axiom of choice for uncountable families of sets; this result is due to R. M. Solovay (see [17], §6). Of course, if $\boldsymbol{B}$ is countable, then 4.1 does not require the axiom of choice.

5. Invariant extensions of measures. Let now G be a group of automorphisms of a Boolean algebra $\boldsymbol{B}$, and $\boldsymbol{R} \subseteq \boldsymbol{B}$ a *G-invariant subring* of $\boldsymbol{B}$, i.e., such a subring that $g(a) \in \boldsymbol{R}$ for every $a \in \boldsymbol{R}$ and $g \in G$. Let h be a homomorphism of G into the multiplicative group of positive reals. Finally, let μ be a measure over $\boldsymbol{R}$ satisfying

$$\mu(g(a)) = h(g)\mu(a) \quad \text{for all } a \in \boldsymbol{R} \text{ and } g \in G.$$

5.1. INVARIANT EXTENSION THEOREM. *If G is amenable, then there exists a measure μ^* extending μ to all of $\boldsymbol{B}$ such that*

$$\mu^*(g(a)) = h(g)\mu^*(a) \quad \textit{for all } a \in \boldsymbol{B} \textit{ and } g \in G. \tag{3}$$

Proof. Let μ' be any extension of μ over $\boldsymbol{B}$ (Theorem 4.1). Let S be the set of all $a \in \boldsymbol{B}$ for which there exists a $b \in \boldsymbol{R}$ with $a \leqslant b$ and $\mu(b) < \infty$. Let us prove the legitimacy of the definition:

$$\mu^*(a) = \begin{cases} \infty & \text{if } a \notin S, \\ \displaystyle\int_G \frac{\mu'(g(a))}{h(g)} m(dg) & \text{if } a \in S, \end{cases}$$

where m is any invariant measure over $P(G)$ with $m(G) = 1$ (it exists because G is amenable). Now, the integral above exists if for each $a \in S$ the function $f(g) = \mu'(g(a))/h(g)$ is bounded. To check this let $b \in \boldsymbol{R}$, $a \leqslant b$ and $\mu(b) < \infty$. Then

$$f(g) = \frac{\mu'(g(a))}{h(g)} \leqslant \frac{\mu'(g(b))}{h(g)} = \frac{\mu(g(b))}{h(g)} = \mu(b),$$

and thus f is bounded.

It remains to check that μ^* is a measure over $\boldsymbol{B}$, that it extends μ and that it satisfies (3). The first two of these properties are obvious and the last is obvious if $a \notin S$. If $a \in S$ and $t \in G$, then

$$\mu^*(t(a)) = \int_G \frac{\mu'(gt(a))}{h(g)} m(dg) = \int_G \frac{\mu'(g(a))}{h(gt^{-1})} m(dg)$$
$$= h(t) \int_G \frac{\mu'(g(a))}{h(g)} m(dg) = h(t)\mu^*(a).$$

5.2. Applications. 1. Let $\boldsymbol{B}$ be the Boolean algebra of all subsets of the plane $\mathscr{R}^2$, G the group of similarities of $\mathscr{R}^2$, $a > 0$, $h(g)$ the a-th power of the magnification factor of g, μ the a-dimensional Hausdorff measure in $\mathscr{R}^2$, and $\boldsymbol{R}$ the algebra of μ-measurable sets. It is well known that G is solvable, and hence amenable. Thus Theorem 5.1 applies, and it solves the problem of Marczewski stated in Section 2.

2. Let $\boldsymbol{B}$ and G be as above, let $h(g) = 1$ for all $g \in G$, $\boldsymbol{R} = \{\emptyset, \mathscr{R}^2\}$, $\mu(\emptyset) = 0$ and $\mu(\mathscr{R}^2) = 1$. Then by 5.1 we get a strange bounded invariant measure over $P(\mathscr{R}^2)$. For a partial extension of this result to $\mathscr{R}^n$ see Section 6.

For other examples see [2].

For supramenable G we have the following

5.3. Theorem. *If G is a supramenable group of automorphisms of a Boolean algebra $\boldsymbol{B}$, and $a \in \boldsymbol{B}$, $a \neq 0$, then there exists a G-invariant measure μ over $\boldsymbol{B}$ (i.e., $\mu(g(x)) = \mu(x)$ for all $g \in G$ and $x \in \boldsymbol{B}$) such that $\mu(a) = 1$.*

Proof. We let S be the Stone space of $\boldsymbol{B}$, and thus we can regard the elements of $\boldsymbol{B}$ as subsets of S and the elements of G as permutations (homeomorphisms) of S. Pick a point $p \in a$. Then let m be a left invariant measure over the algebra of all subsets of G such that

$$m\{g \in G\colon\ g(p) \in a\} = 1;$$

by supramenability of G such a measure m exists. Now for every $x \in \boldsymbol{B}$ we let

$$\mu(x) = m\{g \in G\colon\ g(p) \in x\}.$$

It is clear that μ is a measure over $\boldsymbol{B}$ and that $\mu(a) = 1$. Also, μ is invariant since, by the left invariance of m, for any $f \in G$ we have

$$\begin{aligned}\mu(f(x)) &= m\{g \in G\colon\ g(p) \in f(x)\} = m\{g \in G\colon\ f^{-1}g(p) \in x\}\\ &= m\{fg \in G\colon\ g(p) \in x\} = m(f\{g \in G\colon\ g(p) \in x\})\\ &= m\{g \in G\colon\ g(p) \in x\} = \mu(x).\end{aligned}$$

We have also the following extension theorem without any special assumptions on G.

5.4. Theorem. *If G is a group of automorphisms of a Boolean algebra $\boldsymbol{B}$, $\boldsymbol{I}$ is an ideal in $\boldsymbol{B}$, and m is an invariant measure over $\boldsymbol{I}$ (i.e., if $a \in \boldsymbol{I}$, $g \in G$ and $g(a) \in \boldsymbol{I}$, then $m(a) = m(g(a))$), then there exists an invariant measure μ over $\boldsymbol{B}$ extending m.*

Proof. For every $x \in \boldsymbol{B}$ we put

$$\mu(x) = \sup\Big\{\sum_{i=1}^{n} m(g_i(x_i))\colon\ n < \omega,\ g_i \in G,\ x_i \leqslant x,\ x_i \wedge x_j = 0 \quad \text{for } i \neq j,\ g_i(x_i) \in \boldsymbol{I}\Big\}.$$

All the required properties of μ are visible.

5.5. Remark. A theorem similar to 5.4 is also true for countably additive algebras, ideals and measures (see [17], Theorem 6).

6. The Marczewski measure. Let $\boldsymbol{B}$, G, h, $\boldsymbol{R}$, and μ be as in 5.1. Let $\boldsymbol{I}$ be an ideal in $\boldsymbol{B}$ and assume that $\boldsymbol{I}$ is also G-invariant, i.e., $g(a) \in \boldsymbol{I}$ whenever $a \in \boldsymbol{I}$ and $g \in G$. Moreover, let $\mu(a) = 0$ for all $a \in \boldsymbol{R} \cap \boldsymbol{I}$.

6.1. THEOREM. *If G is amenable, then there exists a measure μ^* satisfying the conclusions of* 5.1 *and, moreover, $\mu^*(a) = 0$ for all $a \in \boldsymbol{I}$.*

Proof. Let $\boldsymbol{R}'$ be the ring generated by $\boldsymbol{R} \cup \boldsymbol{I}$. Then for each $a \in \boldsymbol{R}'$ there exists a decomposition

$$a = r \Delta i, \quad \text{where } r \in \boldsymbol{R} \text{ and } i \in \boldsymbol{I};$$

here Δ is the Boolean symmetric difference (also called the *Boolean addition*). We put $\mu'(a) = \mu(r)$ for all $a \in \boldsymbol{R}'$. It is clear that $\mu'(a)$ does not depend on the choice of r and i. Now the quintuple $\boldsymbol{B}$, G, h, $\boldsymbol{R}'$, μ' satisfies the assumptions of 5.1 and $\mu'(a) = 0$ for all $a \in \boldsymbol{I}$. Thus 5.1 yields 6.1.

6.2. Application. Let $\boldsymbol{B} = P(\mathscr{R}^2)$, let G be the group of all similarities of $\mathscr{R}^2$, $h(g)$ the square of the magnification factor of g, $\boldsymbol{R}$ the ring of subsets of $\mathscr{R}^2$ whose boundaries have 2-dimensional Lebesgue measure 0, μ the Lebesgue measure restricted to $\boldsymbol{R}$, and $\boldsymbol{I}$ the ideal of meager sets in $\mathscr{R}^2$. Then Theorem 6.1 applies and the resulting measure μ^* vanishes on meager sets. As mentioned in Section 2, Marczewski was the first to prove the existence of such measures.

7. What about $\mathscr{R}^n$? Our applications do not extend to $\mathscr{R}^3$, since the group SO(3) of rotations of $\mathscr{R}^3$ around the origin is not amenable. In fact, it has free non-Abelian subgroups (see [4] for the best proof of this fact, and [3], [7] for its "ultimate" refinements). Moreover, the existence of paradoxical decompositions of the surface of a sphere in $\mathscr{R}^3$ makes it impossible to extend those applications. The same is true for $\mathscr{R}^n$ with $n \geqslant 3$. Nothing is known about the possibility of invariant universal extendibility of Hausdorff a-dimensional measures in $\mathscr{R}^n$ when $n \geqslant 3$ and $a < 2$. As it is easily seen, for $a \geqslant 2$ they are not possible for the same reasons as before.

What if we replace universal extendibility (to the algebra of all subsets of $\mathscr{R}^n$) by extendibility to some smaller Boolean algebra, e.g., the algebra $\boldsymbol{L}_n$ of Lebesgue measurable sets in $\mathscr{R}^n$ or the algebra $\boldsymbol{B}_n$ of sets having the property of Baire in $\mathscr{R}^n$? For $\boldsymbol{L}_n$ we have an excellent invariant measure, the Lebesgue n-dimensional measure; but we can still ask whether there exists a bounded measure like that in 5.2.2. The answer is affirmative and it will be the object of our next theorem. As for $\boldsymbol{B}_n$ with

$n \geqslant 3$ nothing is known; this is the problem of Marczewski stated in Section 2 (see [12] and [13]).

Now we shall prove a refinement of Theorem 2 of [16].

7.1. Theorem. *There exists a measure μ over L_n which is invariant under similarities and $\mu(\mathscr{R}^n) = 1$.*

Proof. It is proved in [16] that there exists a measure μ_0 over L_n which satisfies $\mu_0(f[X]) = \mu_0(X)$ for all $X \in L_n$ and all isometries f of $\mathscr{R}^n$. Let G be the group of all magnifications of $\mathscr{R}^n$ from the origin. Then G is Abelian, and hence amenable. Let m be a universal invariant measure over $P(G)$ with $m(G) = 1$. For all $X \in L_n$ we put

$$\mu(X) = \int_G \mu_0(g[X])m(dg),$$

which is permissible since $\mu_0 \leqslant 1$. We claim that μ is a measure satisfying the requirements of 7.1. All is obvious except perhaps the invariance of μ under all similarities of $\mathscr{R}^n$. Evidently, μ is invariant under G. On the other hand, if f is an isometry of $\mathscr{R}^n$, then, for any $g \in G$, gfg^{-1} is also an isometry of $\mathscr{R}^n$. Therefore,

$$\mu_0(gf[X]) = \mu_0(gfg^{-1}g[X]) = \mu_0(g[X]),$$

and hence

$$\mu(gf[X]) = \mu(X).$$

Since every similarity of $\mathscr{R}^n$ is of the form gf, where $g \in G$ and f is an isometry, 7.1 follows.

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Added in proof. For related work on countably additive measures see R. B. Chuaqui, *Measures invariant under a group of transformations*, Pacific Journal of Mathematics 68 (1977), p. 313-329.

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