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BANACH SPACES ISOMORPHIC TO PROPER M-IDEALS

BY

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Introduction. Let S be a closed, bounded, convex subset of a Banach space X. The scalar field may be real or complex, except where it is explicitly specified. If the difference set $S-S$ has an interior point, we will call S a *quasiball*. Then, given a positive integer n, we say that S is an *n-quasiball* if, whenever $x_1, x_2, \ldots, x_n$ are interior points of $\frac{1}{2}(S-S)$, then $\bigcap_{i=1}^{n}(x_i+S)$ is nonempty. These definitions depend only on the topology of X, and remain invariant under any equivalent norm that X might be given. This is the crucial difference between our quasiballs and the pseudoballs defined in [3].

A set S is *symmetric about a point* x if $S-x = x-S$. Any symmetric convex body, in particular a closed ball, is obviously an n-quasiball for all n. To exclude such trivial examples, we will call a quasiball *proper* if it is not symmetric. It is easy to show that every quasiball is a 2-quasiball. What is not at all obvious is that every 3-quasiball is already an n-quasiball for all n. However, this should not surprise the reader familiar with M-ideals.

Let Y be a Banach space containing X (isometrically). We recall that X has the *n-ball property* in Y if, whenever $B_1, \ldots, B_n$ are closed balls in Y, with $X \cap B_i$ nonempty for each i, and $\operatorname{int} \bigcap_{i=1}^{n} B_i$ nonempty, then $X \cap \bigcap_{i=1}^{n} B_i$ is nonempty. It turns out that every subspace with the 3-ball property has already the n-ball property for all n. This result is due to Alfsen and Effros [1], who called such subspaces *M-ideals*. Actually, Alfsen and Effros gave a different definition of M-ideals, and the characterization in terms of balls is a theorem of theirs. For simpler proofs of this, see [2], [9] or [12]. If there is a projection P: $Y \to X$ satisfying the identity

$$\|y\| = \max\{\|Py\|, \|y-Py\|\},$$

then X is said to be an *M-summand* in Y. Elementary calculations show that every M-summand is an M-ideal.

Following [3], we say that X can be a *proper M-ideal* if there is a Banach space Y containing X, so that X is an M-ideal, but not an M-

summand, in Y. It is obvious that any space X is an M-summand in some larger space Y. The standard example of a proper M-ideal is c_0, considered as a subspace of c. Our main result shows that all proper M-ideals are, in a sense, built up from this example.

The relationship between quasiballs and M-ideals depends on some elementary approximation theory, which we develop in the next section. In the main section, we will show that a Banach space can be renormed to be a proper M-ideal if and only if it contains a proper 3-quasiball. (It follows, for example, that any 3-quasiball in l_1, or in a reflexive space, is automatically symmetric.)

Behrends and Harmand [3] showed that every proper M-ideal contains an isomorphic copy of c_0. We will establish an isomorphic converse to this. To be precise, we will show that every Banach space containing c_0 also contains a proper 3-quasiball.

A subspace which has the 2-ball property is called a *semi-M-ideal.* A semi-M-ideal will be called *proper* if it is not an M-summand. (Thus, proper M-ideals are also proper semi-M-ideals.) Examples of semi-M-ideals which are not M-ideals are now well known, at least in real Banach spaces ([1], Theorem 5.9). It is curious that no such example has been found in a complex Banach space. In the last section, we shall make a few remarks about this problem.

This work was done while the author was visiting the Free University of Berlin, and owes much to the stimulation of the work-group there. Indeed, many of the ideas used here appeared originally in [3].

Preliminary results. If A_1 and A_2 are subsets of a Banach space, we will write $A_1 \simeq A_2$ to mean that $\bar{A}_1 = \bar{A}_2$ and $\operatorname{int} A_1 = \operatorname{int} A_2$. Let $U(\cdot)$ denote the closed unit ball of a given space. We say that X is *normed* by a quasiball S if $U(X) \simeq \frac{1}{2}(S-S)$. Obviously, any quasiball in a real Banach space becomes norming, under an equivalent norm for X. If in addition there is a Banach space Y containing X, and a point y in Y such that $S \subset B(y, 1)$, the closed ball with centre y and radius 1, then we say that X is *well-normed* by S. This property is only interesting for complex Banach spaces.

LEMMA 1 ([11], Lemma 5.3). *Let X be a real Banach space, normed by a quasiball S. Then X is well-normed by S.*

It is obvious that any ball of radius one is a well-norming n-quasiball. Note that a norming quasiball which is not a ball cannot be symmetric, and thus is proper. A little approximation theory shows how naturally proper quasiballs arise.

Now, let X be a subspace of Y and fix $y \in Y$. The set of best approximants to y is defined as

$$P(y) = \{x \in X\colon\ \|x-y\| = d(y, X)\}.$$

In general, $P(y)$ may be empty, but this never occurs if X is a semi-M-ideal in Y (see [1], [2], [9] or [12]). In fact, almost the opposite is true.

PROPOSITION 1 ([10], Theorem 1.2). *X is a semi-M-ideal in Y if and only if, for all $y \in Y$ with $d(y, X) = 1$, $P(y)$ is a quasiball which norms X.*

This gives us the fundamental relationship between quasiballs and M-ideals. The next two results are similar.

PROPOSITION 2 ([10], Theorem 1.1). *The following are equivalent:*

(i) *X is an M-ideal in Y.*

(ii) *For all $y \in Y$ with $d(y, X) = 1$, $P(y)$ is a 3-quasiball which norms X.*

(iii) *For all $y \in Y$ with $d(y, X) = 1$ and all $n \in \mathbf{N}$, $P(y)$ is an n-quasiball which norms X.*

PROPOSITION 3 ([10], Theorem 1.3). *X is an M-summand in Y if and only if, for all $y \in Y$ with $d(y, X) = 1$, $P(y)$ is a ball of radius one.*

Quasiballs and containment of c_0. At last, we prove some new results.

PROPOSITION 4. *A Banach space X can be a proper semi-M-ideal if and only if X is well-normed by a proper quasiball.*

Proof. ($\Rightarrow$) This is clear from Propositions 1 and 3.

($\Leftarrow$) Let S be a quasiball norming X, let Y be a space containing X and suppose $S \subset B(y, 1)$ for some y in Y. If $y \in X$, it is easy to show that $S = B(y, 1)$, which is not proper. Thus we may suppose that $Y = X \oplus \mathbf{K}y$. The idea is to renorm Y, whilst preserving the norm on X.

Let U be the closed convex hull of the set

$$\{\lambda(x-y) \colon x \in S, \ |\lambda| \leqslant 1\}.$$

It is clear that U is the unit ball for an equivalent norm on Y, which we shall work with henceforth. The arguments used in [3], Theorem 3.4, show that $U \cap X$ is just the original unit ball of X, and that $S = P(y)$, with respect to this new norm. Since S is not symmetric, Proposition 1 shows that X is a proper semi-M-ideal in Y.

Proposition 4 shows that a real Banach space is isomorphic to a proper semi-M-ideal if and only if it contains a proper quasiball. Since every real Banach space can easily be shown to have both these properties, this equivalence is not very interesting.

THEOREM 1. *A real Banach space X can be a proper M-ideal if and only if it is normed by a proper 3-quasiball.*

Proof. ($\Rightarrow$) This follows from Propositions 2 and 3.

($\Leftarrow$) By Lemma 1, X is well-normed by some proper 3-quasiball S. As before, we can find Y containing X and y in Y so that $P(y) = S$. Now X is a hyperplane in Y and $P(\lambda y + x) = \lambda P(y) + x$ for all $\lambda \in \mathbf{R}$, $x \in X$. Thus $P(z)$ is a 3-quasiball which norms X, whenever $z \in Y$, $d(z, X) = 1$. An application of

Proposition 2 shows that X is an M-ideal in Y. It is proper by Proposition 3.

THEOREM 2. *Let S be a 3-quasiball in some Banach space X. Then S is an n-quasiball for all n.*

Proof. The statement is independent of the scalar field and of the norm, so we may suppose that the scalars are real and that X is normed by S. Once again we construct Y containing X so that $S = P(y)$ for some $y \in Y$, and X is a hyperplane in Y. Another application of Proposition 2 completes this proof.

THEOREM 3. *Let X be a real or complex Banach space, normed by a quasiball S. Then the following are equivalent:*

(i) *S is a 3-quasiball.*

(ii) *S is an n-quasiball for all n.*

(iii) *S is a 5-quasiball.*

(iv) *For all 2-dimensional spaces Y and all quotient maps $Q: X \to Y$, there exists $y_0 \in Y$ with $Q(S) \simeq B(y_0, 1)$.*

(v) *For some $F \in X^{**}$ we have $S \subset B(F, 1)$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). This much is clear.

(iii) $\Rightarrow$ (iv). Consider the family of bounded convex sets $Q(S) - y$, where $\|y\| < 1$. These are the images, under Q, of the sets $S - x$ for $\|x\| < 1$. By hypothesis, every 5 members of this family have a point in common. The dimension of Y over the reals is at most 4, so Helly's theorem gives us a point

$$y_0 \in \bigcap_{\|y\| < 1} (Q(S) - y).$$

Thus $\operatorname{int} B(y_0, 1) \subseteq Q(S)$. Since S has diameter two, this forces

$$Q(S) \simeq B(y_0, 1).$$

(iv) $\Rightarrow$ (v) $\Rightarrow$ (i). These follow from the arguments used in [3], Proposition 3.2.

THEOREM 4. *A complex Banach space can be a proper M-ideal if and only if it is normed by a proper 3-quasiball.*

Proof. Theorem 3 states that a norming 3-quasiball is actually well-norming. The result now follows from the proof already given for real Banach spaces.

It follows from Theorems 1 and 4 that a Banach space is isomorphic to a proper M-ideal if and only if it contains a proper 3-quasiball. It was shown in [3] that every proper M-ideal contains an isomorphic copy of c_0. We are now in a position to prove the converse of this. But first we recall another definition from [3]. Let us say that X has the *intersection property* if, for every $\varepsilon > 0$, one can find finitely many points $x_1, \ldots, x_n$ in $\operatorname{int} U$ such that

$$\bigcap_{i=1}^{n} B(x_i, 1) \subset B(0, \varepsilon).$$

THEOREM. 5. *For any Banach space X, the following are equivalent:*

(i) *X is isomorphic to a proper M-ideal.*

(ii) *X is isomorphic to a space which fails the intersection property.*

(iii) *X contains an isomorphic copy of c_0.*

(iv) *X contains a proper 3-quasiball.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). This is proved in [3], Section 4, using [4], Theorem 5.

(iii) $\Rightarrow$ (iv). Let M be a closed subspace of X, isomorphic to c_0. Then M must contain a proper 3-quasiball S, since c_0 does ([3], Example 2). Note that $S-S$ will only have relative interior points in M.

Let $B=\overline{S+U(X)}$. It is clear that $B-B$ has nonempty interior. A routine application of the separation theorem shows that B is not symmetric. We show that B is a 3-quasiball.

Now, let x_1, x_2, $x_3 \in \operatorname{int}\frac{1}{2}(B-B)$. Then

$$\begin{aligned} x_i &\in \operatorname{int}\tfrac{1}{2}(\overline{S+U}-\overline{S+U}) \\ &\subseteq \operatorname{int}\tfrac{1}{2}\overline{(S+U)-(S+U)} \\ &= \operatorname{int}\left(\overline{\tfrac{1}{2}(S-S)+U}\right) \\ &= \operatorname{int}\left(\tfrac{1}{2}(S-S)+U\right) \quad \text{by [8], Section 22,} \\ &= \operatorname{int}_M\tfrac{1}{2}(S-S)+\operatorname{int} U. \end{aligned}$$

Therefore $x_i=y_i+z_i$, where $y_i \in \operatorname{int}_M\frac{1}{2}(S-S)$ and $z_i \in \operatorname{int} U$. Thus

$$\begin{aligned} \bigcap_{i=1}^{3}(B+x_i) &\supseteq \bigcap_{i=1}^{3}(S+U+y_i+z_i) \\ &\supseteq \bigcap_{i=1}^{3}(S+y_i+0) \quad \text{since } 0 \in U+z_i \\ &\neq \emptyset \quad \text{since } S \text{ is a 3-quasiball in } M. \end{aligned}$$

(iv) $\Rightarrow$ (i). This follows from Theorem 1 or Theorem 4.

If "isomorphic" is replaced by "almost isometric", the resulting version of Theorem 5 is correct. This follows by replacing S by εS in the preceding argument, and the fact that every isomorphic copy of c_0 contains subspaces almost isometric to c_0 ([7], Lemma 2.2).

However, "isomorphic" cannot be replaced by "isometric" in Theorem 5. As noted in [3], every infinite-dimensional $C(K)$-space fails the intersection property, and so cannot be a proper M-ideal. Nonetheless, every such space contains isometric copies of c_0. Conversely, Harmand and Rao [5] have exhibited a smooth space (isomorphic to c_0, in fact) which is an M-ideal in its second dual. Clearly, this space is uncomplemented in its second dual, and contains no isometric copy of c_0.

Complex semi-M-ideals. Let us begin by establishing a version of Lemma 1 which is applicable to complex Banach spaces.

LEMMA 2. *If S is a closed, convex subset of a complex Banach space X, then the following are equivalent*:

(i) *S is a quasiball which well-norms X.*

(ii) *For any norm-one functional $f \in X^*$, there exists a scalar λ such that $f(S) \simeq B(\lambda, 1)$.*

Proof. (i)$\Rightarrow$(ii). Suppose that Y is a Banach space containing X and that $S \subseteq B(y, 1)$ for some $y \in Y$. Given a norm-one functional $f \in X^*$, let $g \in Y^*$ be a norm-preserving extension of f. Then

$$f(S) = g(S) \subseteq B(g(y), 1)$$

(in C). But $\frac{1}{2}(S-S) \simeq B(0, 1)$, so

$$\tfrac{1}{2}(f(S)-f(S)) \simeq B(0, 1).$$

Simple plane geometry then forces $f(S) \simeq B(g(y), 1)$.

(ii)$\Rightarrow$(i). For any norm-one $f \in X^*$, $f(\frac{1}{2}(S-S)) \simeq B(0, 1)$. By the Hahn–Banach theorem and [8] (22.3), S is a quasiball which norms X. As in Lemma 1, we now take $Y = l_\infty(\Gamma)$ for suitable Γ. Fix $f \in Y^*$ with $\|f\| \leqslant 1$. By hypothesis, there is $\lambda \in C$ with $f(S) \subseteq B(\lambda, 1)$. But then

$$\lambda \in \bigcap_{x \in S} B(f(x), 1).$$

Since f was arbitrary, we have shown that the collection of balls $B(x, 1)$, for $x \in S$, has the weak intersection property. Applying this to the evaluation functionals on Y ([6], Lemma 3) we find there exists some $y \in \bigcap_{x \in S} B(x, 1)$. Thus $S \subseteq B(y, 1)$ and S well-norms X.

Lemma 2 is also true, but of little consequence, for real Banach spaces.

It is still unknown whether every semi-M-ideal in a complex Banach space is automatically an M-ideal. We now show that this problem is equivalent to an easily stated problem concerning two-dimensional convex sets. This equivalence is already known [13], but the results developed so far in this paper enable us to give a more transparent proof.

THEOREM 6. *The following statements are either all true or all false*:

(i) *If S is a quasiball which well-norms a complex Banach space, then S is a 3-quasiball.*

(ii) *If S is a compact, convex subset of a finite-dimensional complex vector space X, and $f(S)$ is a disc for every $f \in X^*$, then S is symmetric.*

(iii) *If S is a compact, convex subset of a two-dimensional complex vector space X, and $f(S)$ is a disc for every $f \in X^*$, then S is symmetric.*

(iv) *Every two-dimensional semi-M-ideal in a complex Banach space is already an M-ideal.*

(v) *Every semi-M-ideal in a complex Banach space is already an M-ideal.*

Proof. (i) $\Rightarrow$ (ii). Let X be finite dimensional, and S a compact, convex subset which spans X, and such that, for all $f \in X^*$, $f(S)$ is a disc. We make X^* into a Banach space by defining $\|f\|$ to be the radius of the disc $f(S)$. It is routine to check that this defines a norm on X^*, which then induces a norm on X. Lemma 2 then states that S is a quasiball which well-norms X. By (i), S is a 3-quasiball. Theorem 3 and the reflexivity of X then force $S \subseteq B(x, 1)$ for some $x \in X$. But $\frac{1}{2}(S-S) \simeq B(0, 1)$, so, by the separation theorem, $S = B(x, 1)$ is symmetric.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (iv). Suppose X is a two-dimensional space which is a *proper* semi-M-ideal in some larger space. By Proposition 4, X is well-normed by some proper quasiball S. Obviously, S is compact, and Lemma 2 states that $f(S)$ is a disc for every $f \in X^*$. Then, by our hypothesis, S is symmetric, which is impossible.

(iv) $\Rightarrow$ (v). This is well known, and a proof appears in [13]. It follows easily from the duality theory of M-ideals and semi-M-ideals ([9], Section 6, or [12]).

(v) $\Rightarrow$ (i). Let X be well-normed by a (proper) quasiball S. By Proposition 4, X is a semi-M-ideal in some larger space Y. By (v), X is actually an M-ideal in Y. A careful reading of the proof of Theorem 8 shows that S is a 3-quasiball.

Although statement (iii) appears so much simpler than statement (v), it has stubbornly resisted all attempts to determine its truth or falsity. We hope that the new proof of this equivalence might help to solve this.

PROBLEM. Let S be a compact, convex subset of $\mathbf{C}^2$ and suppose that $f(S)$ is a disc for every linear f: $\mathbf{C}^2 \to \mathbf{C}$. Is S necessarily symmetric? **(P 1354)**

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