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STRONGLY METRIZABLE SPACES OF LARGE DIMENSION EACH SEPARABLE SUBSPACE OF WHICH IS ZERO-DIMENSIONAL

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1. Let us recall that a space S is said to be *strongly metrizable* if it has a base which is the union of countably many star-finite open coverings of S (see [1], Chapter 6, § 3, or [4]). The main goal of this paper is the following

THEOREM. *Let n be an arbitrary natural number. Then there exists a strongly metrizable space S_n with $\dim S_n = n$ such that each separable subspace of S_n is zero-dimensional.*

This Theorem is an immediate corollary to the following

PROPOSITION. *For every metrizable separable space X there exists a strongly metrizable space S with $\dim S = \dim X$ such that every completion of the space S contains a copy of X, but each separable subspace of S is zero-dimensional.*

2. Comments. A metrizable space Y with $\dim Y = 1$ each separable subspace of which is zero-dimensional was constructed by R. Pol (see [5], Example 2). After removing the point 0 from Y we obtain a strongly metrizable space with these properties. It seems that the method of [5] cannot be applied to obtain the spaces of greater dimension; it would be interesting in this context to compute the dimension of the n-th power of Y (**P 1026**). Note also that the metrizable space $\varDelta$ with $\operatorname{ind}\varDelta = 0$ and $\dim\varDelta = 1$, constructed by Roy [7], has also that curiosity; however, $\varDelta$ is not strongly metrizable, since for such spaces the dimensions ind and dim coincide (see [1], Chapter 6, § 3, Theorem 13).

3. Notation and auxiliary result. Our terminology follows [2]. Let $\mathfrak{c}$ denote the power of continuum and let $\mathfrak{c}^+$ be the first cardinal after $\mathfrak{c}$. Let B be the Baire space of weight $\mathfrak{c}^+$, i.e. the Cartesian product of countably many copies of the discrete space of cardinality $\mathfrak{c}^+$ (see [2], Example 4.2.2). Finally, let λ be the initial ordinal of cardinality $\mathfrak{c}$, and λ^+ — the initial ordinal of cardinality $\mathfrak{c}^+$.

We shall use the following result obtained by R. Pol ([6], Section 3).

There exist a decomposition of the space B into disjoint sets $\{B_\xi\colon \xi < \lambda^+\}$ such that

(i) $\bigcup \{B_\alpha\colon \alpha \leqslant \xi\}$ *is closed in B for every $\xi < \lambda^+$,*

and a disjoint decomposition $\{K_\alpha\colon \alpha < \lambda\}$ of the set of ordinals less than λ^+ such that the sets $E_\alpha = \bigcup \{B_\xi\colon \xi \in K_\alpha\}$ have the following property:

(ii) *if $E_\alpha \subset G_\alpha$, where G_α is a G_δ-set in B, then*

$$\bigcap_{\alpha<\lambda} G_\alpha \neq \emptyset.$$

Although the statement was not stated in this form in [6], it is easy to observe that if we apply the reasonings given in the proof of Proposition 3.5 in [6] (with obvious modifications) to the sets constructed in Theorem 3.3 of [6], then we obtain the required result.

4. Proof of the Proposition. Since X is of cardinality not greater than $\mathfrak{c}$, we can choose an injection φ of the set X into the set of ordinals less than λ. Let us put $E_x = E_{\varphi(x)}$ and let

$$S = \bigcup_{x \in X} E_x \times \{x\}$$

be the subspace of the Cartesian product $B \times X$, where B is the Baire space defined above. Since the spaces B and X are strongly metrizable (see [1], Chapter 6, § 3), so is the space S.

We shall verify that if A is a separable subspace of S, then A is zero-dimensional. Let

$$C = \{\xi\colon \xi < \lambda^+ \text{ and } A \cap (B_\xi \times X) \neq \emptyset\}.$$

The set C is countable, for otherwise the sets

$$A_\xi = A \cap \bigcup \{B_\alpha \times X\colon \alpha \leqslant \xi\}, \qquad \xi \in C,$$

would form a well-ordered, uncountable, strictly increasing family of closed (by (i)) subspaces of A, which is impossible (see [3], § 24, II, Theorem 3). Now we infer from the Sum Theorem (see [2], Theorem 7.2.1) that the space $\bigcup_{x \in \varphi^{-1}(C)} E_x \times \{x\}$ is zero-dimensional, as the sets $E_x \times \{x\}$ are closed in S. Since

$$A \subseteq \bigcup_{x \in \varphi^{-1}(C)} E_x \times \{x\},$$

we conclude that $\dim A = 0$.

We shall show that each completion of S contains topologically the space X. By the classical Lavrentiev theorem (see [2], Theorem 4.3.13) it suffices to show that every G_δ-set G in $B \times X$ which contains the space S

contains a copy of X. To this end, let us observe that for each point $x \in X$ the set

$$G_x = \{y \in B\colon (y, x) \in G\}$$

is a G_δ-set in B containing E_x. Thus, by property (ii), there exists a point

$$x_0 \in \bigcap_{x \in X} G_x.$$

Then we have $\{x_0\} \times X \subset G$, which completes the proof.

To prove that $\dim S = \dim X$ let us observe first that

$$\dim S \leqslant \dim(B \times X) \leqslant \dim B + \dim X = \dim X$$

and next that

$$\dim S \geqslant \dim X,$$

as each metrizable space has a dimension-preserving completion (see [2] or [1]).

Remark. It is worthwhile to notice that our reasoning was suggested by a proof of the following simple property of the classical Knaster-Kuratowski space M consisting of the points (x, y), where x runs over the Cantor set C, y belongs to the unit interval I, and y is rational if and only if x belongs to the set Q of the end points of the Cantor set (see [2], P.6.W): every G_δ-set G in $C \times I$ which contains M contains an interval $\{x\} \times I$. We have generalized this construction by taking instead of the decomposition of C into two sets Q and $C \setminus Q$ the decomposition of $B(\mathfrak{c}^+)$ into $\mathfrak{c}$ sets which are non-separable analogues of the classical Bernstein sets in C (cf. [6], Theorem 3.3).

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