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ON COLLECTIONWISE NORMALITY

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1. Introduction. Recent interest in the study of metacompact spaces has led to generalizations such as θ-refinability of Worrell and Wicke [16], weak $\bar{\theta}$-refinability of Smith [10], and weak θ-refinability of Bennett and Lutzer [1]. These notions have been shown to play important roles in the study of paracompact, collectionwise normal and irreducible spaces. For example, in [3] and [10] it has been shown that weak $\bar{\theta}$-refinable spaces are irreducible.

Definition. A space X is called *weak $\bar{\theta}$-refinable* if every open cover of X has a refinement $\bigcup_{i=1}^{\infty} \mathscr{G}_i$ satisfying

(i) each $\mathscr{G}_i = \{G(a, i)\colon a \in A_i\}$ is a collection of open subsets of X;

(ii) for each $x \in X$, $0 < \operatorname{ord}(x, \mathscr{G}_k) < \infty$ for some k;

(iii) the open cover $\{G_i^* = \bigcup \{G\colon G \in \mathscr{G}_i\}\}_{i=1}^{\infty}$ is point finite.

An open cover which satisfies properties (i)-(iii) will be called a *weak $\bar{\theta}$-cover*.

In [9] the author has shown the following

THEOREM 1.1. *Let X be a normal space. Then X is collectionwise normal iff every weak $\bar{\theta}$-cover of X has a locally finite open refinement.*

This somewhat unusual open cover characterization of collectionwise normality leads to some natural questions. In particular, which characterizations for paracompactness provide analogous ones for collectionwise normality? In this paper we consider those characterizations given by Katuta [4], Michael [5]-[7], and Vaughan [15]. These are summarized in the next theorem.

THEOREM 1.2. *Let X be a regular space. Then the following conditions are equivalent*:

(1) *X is paracompact.*

(2) *Every open cover of X has a locally finite refinement.*

(3) *Every open cover of X has a locally finite closed refinement.*

(4) *Every open cover of X has a σ-locally finite open refinement.*

(5) *Every open cover of X has a closure-preserving refinement.*

(6) *Every open cover of X has a σ-closure-preserving refinement.*

(7) *Every open cover of X has a cushioned refinement.*

(8) *Every open cover of X has a σ-cushioned open refinement.*

(9) *Every open cover of X has a linearly locally finite open refinement.*

(10) *Every open cover of X has a linearly closure-preserving open refinement.*

(11) *Every open cover of X has a linearly cushioned open refinement.*

(12) *Every open cover of X has an order locally finite open refinement.*

(13) *Every open cover of X has a well-ordered closure-preserving open refinement.*

(14) *Every open cover of X has a well-ordered cushioned open refinement.*

Remark. Michael has shown (2)-(8) in [5]-[7], Vaughan (9)-(11), (13), and (14) in [15], while Katuta proved (12) in [4].

In Section 2 of this paper we obtain the characterizations analogous to those of Michael for collectionwise normal spaces. We then show several applications of such characterizations. Characterizations by way of linearly locally finite refinements are obtained in Section 3. Order locally finiteness is discussed in Section 4 with conditions (13) and (14) weakened. Several open questions are also provided.

The next theorem of Zenor [17] is very useful in obtaining the results which follow.

THEOREM 1.3. *A space X is collectionwise normal iff for each discrete collection $\{F_\alpha: \alpha \in A\}$ of closed sets there exists a sequence of collections $\{V(\alpha, i): \alpha \in A\}_{i=1}^{\infty}$ of open subsets of X satisfying*

(i) *$\{V(\alpha, i)\}_{i=1}^{\infty}$ covers F_α for each $\alpha \in A$;*

(ii) *$F_\alpha \cap [\bigcup_{\beta \neq \alpha} V(\beta, i)]^- = \emptyset$ for each $\alpha \in A$ and each i.*

2. Some characterizations and applications.

Definition. (1) A collection $\{H_\alpha: \alpha \in A\}$ is called *locally finite* in a space X if every $x \in X$ has a neighborhood which intersects only finitely many H_α. A collection

$$\mathscr{H} = \bigcup_{i=1}^{\infty} \mathscr{H}_i$$

is *σ-locally finite* if each $\mathscr{H}_i$ is a locally finite collection.

(2) A space X is called *discretely expandable* if for every discrete collection $\{F_\alpha: \alpha \in A\}$ there exists a locally finite collection $\{G_\alpha: \alpha \in A\}$ of open sets such that $F_\alpha \subseteq G_\alpha$ for each $\alpha \in A$.

THEOREM 2.1. *Let X be a normal space. Then the following conditions are equivalent:*

(1) *X is collectionwise normal.*

(2) *Every weak $\bar{\theta}$-cover of X has a locally finite open refinement.*

(3) *Every weak $\bar{\theta}$-cover of X has a locally finite closed refinement.*

(4) *Every weak $\bar{\theta}$-cover of X has a locally finite refinement.*

(5) *Every weak $\bar{\theta}$-cover of X has a σ-locally finite open refinement.*

Proof. Clearly, (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4). Since X is normal, every point finite open cover is shrinkable so that (2) $\Rightarrow$ (3) as well.

Part I. We show that (4) $\Rightarrow$ (1). Let $\{F_\alpha \colon \alpha \in A\}$ be a discrete collection of closed subsets of X. Since X is normal, for each $\alpha \in A$ there exists an open set G_α such that

(i) $$F_\alpha \subseteq G_\alpha,$$

(ii) $$\bar{G}_\alpha \cap \bigcup_{\beta \neq \alpha} F_\beta = \emptyset.$$

Let G_0 be an open set such that

$$\bigcup_{\alpha \in A} F_\alpha \subseteq G_0 \subseteq \bar{G}_0 \subseteq \bigcup_{\alpha \in A} G_\alpha$$

and put $\mathscr{G}_1 = \{G_\alpha \colon \alpha \in A\}$ and $\mathscr{G}_2 = \{X - \bar{G}_0\}$. Then $\mathscr{G} = \mathscr{G}_1 \cup \mathscr{G}_2$ is a weak $\bar{\theta}$-cover of X. By (4), $\mathscr{G}$ has a locally finite refinement $\mathscr{H} = \{H_\alpha \colon \alpha \in A\} \cup \{H_0\}$, where $H_0 \subseteq X - \bar{G}_0$ and $H_\alpha \subseteq G_\alpha$ for each $\alpha \in A$. Since each member of $\mathscr{G}$ intersects at most one F_α, $F_\alpha \subseteq H_\alpha$ for each $\alpha \in A$. Furthermore, $\overline{H_\beta} \cap F_\alpha = \emptyset$ for $\beta \neq \alpha$. Write

$$U_\alpha = X - [(\bigcup_{\substack{\beta \in A \\ \beta \neq \alpha}} \overline{H_\beta}) \cup \overline{H_0}] \quad \text{for each } \alpha \in A.$$

Then $F_\alpha \subseteq U_\alpha \subseteq H_\alpha$ so that $\{U_\alpha \colon \alpha \in A\}$ is a locally finite open collection. Therefore, X is discretely expandable, and hence collectionwise normal by Theorem 2.2 of [12].

Part II. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (5) so that we need only to show (5) $\Rightarrow$ (1). Let $\{F_\alpha \colon \alpha \in A\}$ be a discrete collection of closed subsets of X and let $\mathscr{G}$ be the weak $\bar{\theta}$-cover defined in Part I. Then $\mathscr{G}$ has a σ-locally finite open refinement

$$\mathscr{V} = \bigcup_{i=1}^{\infty} \mathscr{V}_i.$$

But $\mathscr{V}$ satisfies conditions (i) and (ii) of Theorem 1.3, so that X is collectionwise normal.

COROLLARY 2.1 (Smith). *A space X is paracompact iff X is weak $\bar{\theta}$-refinable and collectionwise normal.*

COROLLARY 2.2 (Michael). *The Countable Sum Theorem and the Locally Finite Sum Theorem hold for collectionwise normality.*

Definition. (1) A collection $\{H_a\colon a \in A\}$ is *closure-preserving* if, for every $B \subseteq A$,

$$\bigcup_{\beta \in B} \overline{H_\beta} = \overline{\bigcup_{\beta \in B} H_\beta}.$$

(2) A collection $\mathscr{U}$ *is cushioned in a collection* $\mathscr{V}$ with cushion map $f\colon \mathscr{U} \to \mathscr{V}$ if, for every subcollection $\mathscr{U}'$ of $\mathscr{U}$,

$$\bigcup \{U\colon U \in \mathscr{U}'\}^- \subseteq \bigcup \{f(U)\colon U \in \mathscr{U}'\}.$$

The notions of σ-closure-preserving and σ-cushioned should be obvious.

THEOREM 2.2. *Let X be a normal space. Then the following conditions are equivalent:*

(1) *X is collectionwise normal.*

(2) *Every weak $\bar{\theta}$-cover of X has a closure-preserving refinement.*

(3) *Every weak $\bar{\theta}$-cover of X has a cushioned refinement.*

(4) *Every weak $\bar{\theta}$-cover of X has a σ-closure-preserving open refinement.*

(5) *Every weak $\bar{\theta}$-cover of X has a σ-cushioned open refinement.*

Proof. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) from Theorem 2.1. Also (1) $\Rightarrow$ (4) $\Rightarrow$ (5). In [7] Michael showed that every open cover which has a σ-cushioned open refinement has a cushioned refinement, so that (5) $\Rightarrow$ (3). We now show that (3) $\Rightarrow$ (1). Let $\{F_a\colon a \in A\}$ be a discrete collection of closed subsets of X and let $\mathscr{G} = \{G_a\colon a \in A \cup \{0\}\}$ be the weak $\bar{\theta}$-cover defined in the proof of Theorem 2.1. By the same argument as used in the proof of Lemma 3.3 of [7], for each i there exists a cushioned refinement $\{C(a, i)\colon a \in A \cup \{0\}\}$ of $\mathscr{G}$ such that

(a) $$\Big[\bigcup_{\beta < a} C(\beta, i)\Big]^- \cap C(a, i+1) = \emptyset,$$

(b) $$C(a, i) \cap \overline{\Big[\bigcup_{\beta > a} C(\beta, i+1)\Big]} = \emptyset \quad \text{for all } a \text{ and } i.$$

This is possible since at each stage the given cover to be refined is a weak $\bar{\theta}$-cover.

Therefore, there exists a σ-disjoint open refinement

$$\{V(a, i)\colon a \in A \cup \{0\}\}_{i=1}^{\infty}$$

of $\mathscr{G}$ such that $V(a, i) \subseteq C(a, i) \subseteq G_a$ (see Lemma 3.4 of [7]).

Since $\mathscr{V}_i = \{V(a, i)\colon a \in A\}$ is cushioned in $\mathscr{G}$ for each i, X is collectionwise normal by Theorem 1.3.

COROLLARY 2.3. *The closed continuous image of a collectionwise normal space is collectionwise normal.*

Proof. Let $f\colon X \to Y$ be continuous and closed, and let X be collectionwise normal. If $\mathscr{U}$ is a weak $\bar{\theta}$-cover of Y, then $\bar{f}(\mathscr{U})$ is a weak $\bar{\theta}$-cover of X, and hence has a closure-preserving refinement $\mathscr{V}$. But $f(\mathscr{V})$ is a closure-preserving refinement of $\mathscr{U}$, since f is closed. Thus Y is collectionwise normal by Theorem 2.2.

3. Linearly locally finite refinements. In [15] Vaughan studied the properties of linearly locally finite collections and natural generalizations.

Definition. Let $\mathscr{U} = \{U_a: a \in A\}$ be a collection of subsets of a space X and let $\leqslant$ be a linear order on A.

(1) The collection $\mathscr{U}$ is *linearly locally finite* with respect to $\leqslant$ provided every majorized subcollection of $\mathscr{U}$ is locally finite.

(2) The collection $\mathscr{U}$ is *linearly closure-preserving* with respect to $\leqslant$ provided every majorized subcollection of $\mathscr{U}$ is closure-preserving.

(3) The collection $\mathscr{U}$ is *linearly cushioned in a collection* $\mathscr{V}$ with cushioned map f: $\mathscr{U} \to \mathscr{V}$ provided for every majorized subcollection $\mathscr{U}'$ of $\mathscr{U}$ we have

$$\bigcup \{U: U \in \mathscr{U}'\}^- \subseteq \{f(U): U \in \mathscr{U}'\}.$$

THEOREM 3.1. *Let X be a normal space. Then the following conditions are equivalent:*

(1) *X is collectionwise normal.*

(2) *Every weak $\bar{\theta}$-cover of X has a linearly locally finite open refinement.*

(3) *Every weak $\bar{\theta}$-cover of X has a linearly closure-preserving open refinement.*

(4) *Every weak $\bar{\theta}$-cover of X has a linearly cushioned open refinement.*

Proof. Clearly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4); so assume (4). Let $\mathscr{G}$ be any weak $\bar{\theta}$-cover of X. Then $\mathscr{G}$ has a linearly cushioned open refinement $\mathscr{U}$ with respect to some linear order $\leqslant$. Let f: $\mathscr{U} \to \mathscr{G}$ be the associated cushion map. By Lemma 1 of [15] we may assume that $\leqslant$ is a well-order. Write

$$W_U = U - \bigcup \{U' \in \mathscr{U}: U' < U\} \quad \text{for each } U \in \mathscr{U}.$$

We assert that $\mathscr{W} = \{\mathscr{W}_U: U \in \mathscr{U}\}$ is a cushioned refinement in $\mathscr{G}$. Clearly, $\mathscr{W}$ refines $\mathscr{U}$ and is cushioned in $\mathscr{G}$ by the cushioned map g: $\mathscr{W} \to \mathscr{G}$ defined by $g(W_U) = f(U)$. Also $\mathscr{W}$ covers X, for if U is the first member of $\mathscr{U}$ which contains x, then $x \in W_U$. Therefore, X is collectionwise normal by Theorem 2.2.

4. Order locally finiteness. Katuta [4] showed that a space X is paracompact iff every open cover of X has an order locally finite open refinement. Vaughan [15] and Singal and Arya [8] have obtained analogous characterizations using the notions of well-ordered closure-preserving and well-ordered cushioned refinements. It was shown in [15] that the corresponding results were not true if the well-ordered condition was weakened to linearly ordered, even if the refinements were precise or one-to-one. That is, $\mathscr{V}$ is a *precise refinement* of $\mathscr{U} = \{U_\alpha\colon \alpha \in A\}$ if $\mathscr{V} = \{V_\alpha\colon \alpha \in A\}$, where $V_\alpha \subseteq U_\alpha$ for each $\alpha \in A$. Strangely enough, such characterizations are true for collectionwise normality.

Definition. Let $\mathscr{U} = \{U_\alpha\colon \alpha \in A\}$ be a collection of subsets of a space X and let $\leqslant$ be a linear order on A.

(1) The collection $\mathscr{U}$ is *order locally finite* with respect to $\leqslant$ provided for every $U_\alpha \in \mathscr{U}$ the collection $\{U_\beta\colon \beta \leqslant \alpha\}$ is locally finite at each point of U_α.

(2) The collection $\mathscr{U}$ is *order closure-preserving* provided, for every $U_\alpha \in \mathscr{U}$, every $x \in U_\alpha$ and every $B \subseteq \{\beta\colon \beta \leqslant \alpha\}$, if $x \in \overline{\bigcup_{\beta \in B} U_\beta}$, then

$$x \in \overline{U_\beta} \quad \text{for some } \beta \in B.$$

(3) The collection $\mathscr{U}$ is *order-cushioned* in $\mathscr{V}$ with cushion map $f\colon \mathscr{U} \to \mathscr{V}$ provided, for every $U_\alpha \in \mathscr{U}$, every $x \in U_\alpha$ and every $B \subseteq \{\beta\colon \beta \leqslant \alpha\}$, if $x \in \overline{\bigcup_{\beta \in B} U_\beta}$, then

$$x \in \bigcup_{\beta \in B} f(U_\beta).$$

THEOREM 4.1. *Let X be a normal space. Then the following conditions are equivalent:*

(1) *X is collectionwise normal.*

(2) *Every weak $\bar\theta$-cover of X has an order locally finite open refinement.*

(3) *Every weak $\bar\theta$-cover of X has a precise order closure-preserving open refinement.*

(4) *Every weak $\bar\theta$-cover of X has a precise order-cushioned open refinement.*

(5) *Every weak $\bar\theta$-cover of X has a well-ordered closure-preserving open refinement.*

(6) *Every weak $\bar\theta$-cover of X has a well-ordered cushioned open refinement.*

Proof. The fact that (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follows in the same manner as Theorem 2 of [15]. Also (1) $\Rightarrow$ (3) $\Rightarrow$ (4), so we will show that (4) $\Rightarrow$ (1). Let $\{F_\alpha\colon \alpha \in A\}$ be any discrete collection of closed subsets of X and choose $\mathscr{G}$ to be the weak $\bar\theta$-cover defined in the proof of Theorem 2.1. Then $\mathscr{G}$ has a precise order-cushioned open refinement

$$\mathscr{U} = \{U_\alpha\colon \alpha \in A\} \cup \{U_0\}.$$

Now write $W_U = U - \bigcup\{V\colon V < U\}$ for each $U \in \{U_\alpha\colon \alpha \in A\}$, where $\leqslant$ is the linear order on $\mathscr{U}$. It is easy to show that $\mathscr{W} = \{W_U\colon U \in \mathscr{U}\}$

is cushioned in $\mathscr{U}$. Furthermore, $\mathscr{W}$ covers $\bigcup_{a\in A} F_a$; since, for $x \in \bigcup_{a\in A} F_a$, x belongs to F_{a_0} only. Hence

$$x \in W_{U_{a_0}} = U_{a_0} - \bigcup \{V: V < U_{a_0}\}.$$

Therefore, $\mathscr{U}^* = \mathscr{W} \cup \{U_0\}$ is a cushioned refinement of $\mathscr{G}$, so that X is collectionwise normal by Theorem 2.2.

QUESTIONS. (1) Can either the condition "precise" or "well-ordered" be removed in Theorem 4.1? (**P 1043**)

Remark. Vaughan [15] has pointed out that the non-paracompact space $[0, \Omega)$ with the order topology has the property that every open cover has an order closure-preserving open refinement. This space is collectionwise normal however.

(2) Do the open cover characterizations in this paper provide new characterizations for collectionwise normality by way of products? (**P 1044**) (See [13] and [14].)

(3) Do nice characterizations still exist if the space is not assumed to be normal? (**P 1045**)

(4) Is collectionwise normality equivalent to every weak $\bar{\theta}$-cover being shrinkable? (**P 1046**)

REFERENCES

[1] H. Bennett and D. J. Lutzer, *A note on weak θ-refinability*, General Topology and its Applications 2 (1972), p. 49-54.

[2] J. Boone, *On irreducible spaces*, Bulletin of the Australian Mathematical Society 12 (1975), p. 143-148.

[3] — *On irreducible spaces, II*, Pacific Journal of Mathematics 62 (1976), p. 351-357.

[4] Y. Katuta, *A theorem on paracompactness of product spaces*, Proceedings of the Japan Academy 43 (1967), p. 615-618.

[5] E. Michael, *A note on paracompact spaces*, Proceedings of the American Mathematical Society 4 (1953), p. 831-838.

[6] — *Another note on paracompact spaces*, ibidem 8 (1957), p. 822-828.

[7] — *Yet another note on paracompact spaces*, ibidem 10 (1959), p. 309-314.

[8] M. Singal and Shashi Prabha Arya, *A note on order paracompactness*, Bulletin of the Australian Mathematical Society 4 (1971), p. 273-279.

[9] J. Smith, *New characterizations for collectionwise normal spaces*, Glasnik Matematički (to appear).

[10] — *Properties of $\bar{\theta}$-refinable spaces*, Proceedings of the American Mathematical Society 53 (1975), p. 511-517.

[11] — *A remark on irreducible spaces*, ibidem 57 (1976), p. 133-139.

[12] — and L. Krajewski, *Expandability and collectionwise normality*, Transactions of the American Mathematical Society 160 (1971), p. 437-451.

[13] H. Tamano, *Note on paracompactness*, Journal of Mathematics of Kyoto University 3 (1963), p. 137-143.

[14] — *On some characterizations of paracompactness*, Topology Conference, Arizona State University, Tempe, Arizona, 1967, p. 277-285.

[15] J. Vaughan, *Linearly ordered collections and paracompactness*, Proceedings of the American Mathematical Society 147 (1970), p. 186-192.

[16] J. Worrell and H. Wicke, *Characterizations of developable spaces*, Canadian Journal of Mathematics 17 (1965), p. 820-830.

[17] P. Zenor, *Some continuous separation axioms*, Fundamenta Mathematicae 90 (1975-1976), p. 143-158.

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