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EXTENDING COMPLETE CONTINUOUS PSEUDOMETRICS

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In [1] (p. 20) the claim is made that if S is a subspace of a topological space X, then every complete, totally bounded continuous pseudometric on S can be extended to a continuous pseudometric on X. If we let X be the unit interval, $S = [0,1]-\{1/2\}$ and define f on S by $f(x) = 0$ if $x < 1/2$ and by $f(x) = 1$ if $x > 1/2$, then the continuous pseudometric Ψ_f defined by $\Psi_f(x, y) = |f(x)-f(y)|$ is complete and totally bounded. If it extended to a continuous pseudometric d on X, there would exist an open interval (a, b) containing $1/2$ and contained in the d-sphere about $1/2$ with radius $1/2$. Choosing $a < x < 1/2$ and $1/2 < y < b$, we have $d(x, y) < 1$, but $d(x, y) = \Psi_f(x, y) = 1$. In this note we give the correct formulation of this result, namely that the extendability of every complete, totally bounded pseudometric is equivalent to C^*-embedding.

Recall that if γ is an infinite cardinal number, then S is *P^γ-embedded* (*P-embedded*) in X if every continuous γ-separable (continuous) pseudometric on S extends to a continuous pseudometric on X. See [3] for information on this concept. P-embedding becomes an important concept for studying the extendability of various types of functions. For example, in [2] Alò and Sennott showed that S is P-embedded in X iff for all compact T_2-spaces Y we have $S\times Y$ C^*-embedded in $X\times Y$. More recently, Morita and Hoshina [6] refined this and other results and inferred (among other results) that if Y is a compact T_2-space of weight γ, then S is P^γ-embedded in X iff $S\times Y$ is C^*-embedded in $X\times Y$. (This was also obtained in [7].) They also stated [5] that S is P^γ-embedded in X iff (X, S) has the Homotopy Extension Property with respect to every complete ANR space of weight less than or equal to γ. In this note we will show that S is P^γ-embedded in X iff every complete continuous γ-separable pseudometric on S extends to a continuous pseudometric on X. Several consequences of this result will be discussed.

For the remainder of this note, S will always represent a subspace of a topological space X, γ will denote an infinite cardinal number, and all functions and pseudometrics will be assumed continuous.

The following proposition shows how complete pseudometrics can be formed.

PROPOSITION 1. *Let f be a function from X to the pseudometric space (Y, m). Then the pseudometric d defined by $d(x, y) = m(f(x), f(y))$ is complete iff $f(X)$ is a complete subspace of Y.*

COROLLARY. *Let f be a real-valued function on X. Then Ψ_f is complete iff $f(X)$ is closed; Ψ_f is complete and totally bounded iff $f(X)$ is compact.*

The following lemma will be useful several times.

LEMMA 1. *Let S be a subspace of X and let f be a real-valued function on S. Then f extends to a function on X iff Ψ_f extends to a pseudometric on X.*

Proof. If f extends to f^* on X, then Ψ_{f^*} is an extension of Ψ_f. Now assume that Ψ_f extends to a pseudometric d on X. Then f is uniformly continuous with respect to d, hence f extends to a function on the closure of S in (X, d). Since (X, d) is normal, f extends to a function on X.

For a direct proof (not using the C-embedding of closed subsets of normal spaces) of a slightly different result, see 15.1 and 15.2 of [3].

THEOREM 1. *Let S be a subspace of a topological space X. The following are equivalent:*

(1) *S is C^*-embedded in X.*

(2) *Every complete, totally bounded pseudometric on S extends to a pseudometric on X.*

(3) *Every function f on S that is either 2-valued or whose image is a closed interval extends to a function on X.*

Proof. It is known (17.10 of [3]) that S is C^*-embedded in X iff every totally bounded pseudometric on S extends to a pseudometric on X. Hence it is clear that (1) implies (2). To show that (2) implies (3), observe that if f is a function as in (3), then Ψ_f is complete and totally bounded, hence extends to a pseudometric d on X. By Lemma 1, f extends to X. To show that (3) implies (1), it is sufficient to prove that any two completely separated sets in S are completely separated in X ([4], p. 18). Let A and B be subsets of S and let f be a function from S into $[0, 1]$ such that $f(A) \equiv 0$ and $f(B) \equiv 1$. If $f(S) = [0, 1]$, then f extends to f^* on X that separates A and B. Assume that there exists an r, $0 < r < 1$, such that $r \notin f(S)$. Then g, defined by $g(x) = 0$ if $f(x) < r$ and by $g(x) = 1$ if $f(x) > r$, is 2-valued, hence extends to g^* on X and g^* separates A and B.

In [9] an example is given of a zero-dimensional space X having a dense, non-C^*-embedded subset S such that every 2-valued function on S extends to X. Hence, even in the presence of a strong type of disconnectedness, it is still necessary to consider functions whose ranges are closed intervals in showing the C^*-embedding.

Using similar reasoning to that of Theorem 1, we obtain

COROLLARY. *Let S be a subspace of a topological space X. The following are equivalent*:

(1) *S is C-embedded in X.*

(2) *Every complete separable pseudometric on S extends to a pseudometric on X.*

(3) *Every function f on S such that $f(S)$ is either discrete in R or a closed interval* (*bounded or unbounded*) *extends to a function on X.*

Proof. It is known (16.4 of [3]) that S is C-embedded in X iff every separable pseudometric on S extends to a pseudometric on X. Hence it is clear that (1) implies (2). f is as in (3), then Ψ_f is complete and separable, hence extends to a pseudometric d on X. By Lemma 1, f extends to X. To show that (3) implies (1), observe that, by Theorem 1, S is C^*-embedded in X. It is only necessary to show that every zero set disjoint from S is completely separated from S ([4], p. 19). Let $Z(f)\cap S = \emptyset$. We may assume that $0 \leqslant f \leqslant 1$; hence $f|S$ is positive and $g = 1/f$ is greater than or equal to 1 on S. If g is bounded on S, it extends to $\hat{g}$ on X and $\hat{g}f$ separates $Z(f)$ and S. Assume that g is unbounded on S. If there exists a number r such that $[r, +\infty) \subset g(S)$, define h on S by $h = g\vee r$. (By $f\vee g$ ($f\wedge g$) we mean the supremum (infimum) of the functions f and g.) Then, by (3), h extends to $\hat{h}$ on X and $\hat{h}f\wedge 1$ separates $Z(f)$ and S. Suppose that there exists an unbounded, increasing, discrete sequence of numbers (r_n) such that $r_n \notin g(S)$ for all n. Assume that $r_1 \geqslant 1$. Define h on S as follows: $h(x) = r_1$ if $g(x) < r_1$, and $h(x) = r_n$ if $r_{n-1} < g(x) < r_n$. Then h is continuous and $h(S)$ is discrete, hence h extends to $\hat{h}$ on X and $\hat{h}f\wedge 1$ separates $Z(f)$ and S.

To show that P^γ-embedding is equivalent to requiring every complete γ-separable pseudometric to extend, we need to use the space $J(\gamma)$, the "hedgehog with γ spikes". This space is formed by taking γ disjoint copies of the unit interval, identifying the point 0 of each interval and defining a metric m as follows: $m(r, s) = |r-s|$ if r and s are in the same spike, and $m(r, s) = r+s$ otherwise. Under the metric m, the hedgehog is a complete γ-separable metrizable AE for metric spaces.

THEOREM 2. *Let S be a subspace of a topological space X. Then S is P^γ-embedded* (*P-embedded*) *in X iff every complete γ-separable* (*complete*) *pseudometric on S extends to a pseudometric on X.*

Proof. Since S is P-embedded in X iff it is P^γ-embedded in X for all infinite cardinal numbers γ, it is sufficient to show the first statement. The necessity is clear. To show the sufficiency we make use of a theorem of Przymusiński [7] which states that S is P^γ-embedded in X iff every function from S into $J(\gamma)$ extends to a function on X. We prove the condition

(*) Given a function f from a space Y into $J(\gamma)$, and a natural number n, there exists a function g_n from Y into $J(\gamma)$ such that $g_n(Y)$ is closed in $J(\gamma)$ and $m(g_n(x), f(x)) \leqslant 1/n$ for all $x \in Y$.

This will complete the proof. In fact, if f is a function from S into $J(\gamma)$ and g_n is as in (*) for every n, then $d_n(x, y) = m(g_n(x), g_n(y))$ is a complete γ-separable pseudometric on S, hence extends to a pseudometric d_n^* on X. Letting $\mathscr{D}$ be the pseudometric uniformity generated by $\{d_n^*: n \in N\}$, we see that f is uniformly continuous with respect to $\mathscr{D}$, hence extends to the closure of S in $(X, \mathscr{D})$. Since $J(\gamma)$ is an AE for metric spaces, f extends to a function on X.

To verify (*), let us denote the spikes of $J(\gamma)$ by I_a with points (t, a), $0 \leqslant t \leqslant 1$. Consider

$$I_a^i = [(i/n, a), ((i+1)/n, a)], \quad \text{where } 0 \leqslant i < n.$$

Let $i \geqslant 1$ for the moment. If $f(Y) \not\supset I_a^i$, choose $(t_a^i, a) \in I_a^i - f(Y)$ and fix it. We drop the indices in what follows to make the definition of g_n less cumbersome.

If $f(x) \in I^i$ for some $1 \leqslant i < n$ and $f(Y) \supset I^i$, then $g_n(x) = f(x)$. For $f(Y) \not\supset I^i$, let $g_n(x) = (i+1)/n$ if $f(x) > t^i$ and $g_n(x) = i/n$ if $f(x) < t^i$.

If $0 \notin f(Y)$ and $f(x) \in I^0$, let $g_n(x) = 1/n$. If $0 \in f(Y)$ and $f(x) \in I^0$, consider two cases:

1. If $f(Y) \supset [1/2n, 1/n]$, let h be the map of $(1/2n, 1/n]$ onto $(0, 1/n]$ defined by $h(x) = 2x - 1/n$. If $f(x) \in (1/2n, 1/n]$, let $g_n(x) = h(f(x))$, otherwise let $g_n(x) = 0$.

2. If $f(Y) \not\supset [1/2n, 1/n]$, choose a point $t^0 \in [1/2n, 1/n] - f(Y)$ and let $g_n(x) = 1/n$ for $f(x) > t^0$, otherwise $g_n(x) = 0$.

This completes the definition of g_n. Clearly, $g_n(Y)$ is closed in $J(\gamma)$ and no point is moved more than $1/n$.

COROLLARY. *S is P^γ-embedded in X iff every function from S onto a closed subset of a complete metrizable AE (metric) Y with weight of Y less than or equal to γ can be extended over X.*

Proof. The necessity is shown in [5] (and also, independently, in [7]). To show the sufficiency, observe by the proof of Theorem 2 that the extendability of each g_n guarantees the extendability of d. But each g_n is a function from S onto a closed subset of a complete metric AE with weight less than or equal to γ.

In [8] we introduced the notion of M^γ-embedding: S is *M^γ-embedded* (*M-embedded*) in X if every function from S to a metrizable γ-separable (metrizable) AE (metric) extends to X. This notion was characterized and the following proposition given:

If (1) *every γ-separable pseudometric on S is majorized by a complete γ-separable pseudometric on S, and* (2) *S is P^γ-embedded in X, then S is M^γ-embedded in X.*

If every complete γ-separable pseudometric on S extends and (1) holds, then S is clearly P^{γ}-embedded in X. Theorem 2 shows that this is true without assumption (1).

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