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ON QUASI-DOMINATION OF COMPACTA

BY

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1. Introduction. The notions of U-domination, quasi-domination, quasi-affinity, and quasi-equivalence of compacta have been introduced by Borsuk in [3]. These relations are weaker than the relations of fundamental domination and of fundamental equivalence and they allow us to consider shapes from a quantitative point of view.

In the present note we study several properties of quasi-domination in connection with some shape invariants and with the concept of X-likeness (due to Mardešić and Segal [10], p. 146). Some of our theorems generalize results of Borsuk.

We assume that the basic notions and the most elementary results of the theory of shape are known to the reader, who can find them in [2], [7], and [9]. We recall now the concept of quasi-domination (cf. [8]).

Let X and Y be compacta (i.e., compact metrizable spaces). We say that Y is *quasi-dominated* by X provided that for every map h: $Y \to P$, where P is a polyhedron, there are shape morphisms $\boldsymbol{f}$: $X \to Y$ and $\boldsymbol{g}$: $Y \to X$ such that

$$S(h)\cdot \boldsymbol{f}\cdot \boldsymbol{g} = S(h),$$

where $S(h)$ is the shape morphism induced by h.

The former definition is equivalent to the original Borsuk's definition in [3]. If we suppose that X and Y lie in the Hilbert cube Q, it is possible to characterize quasi-domination as follows:

Y is quasi-dominated by X if and only if for every closed neighborhood V of Y in Q there exist shape morphism $\boldsymbol{f}$: $X \to Y$ and $\boldsymbol{g}$: $Y \to X$ such that

$$S(i)\cdot \boldsymbol{f}\cdot \boldsymbol{g} = S(i),$$

where i: $Y \to V$ is the inclusion.

We shall use the notation $Y \leqslant^q X$ to express the fact that Y is quasi-dominated by X. Some properties of quasi-domination can be found in [3], [4], [6], [8], and [12].

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useful conversations and to the referee for valuable suggestions, especially concerning the preceding formulations of quasi-domination.

2. Quasi-domination and movable compacta. The main result of this section is Theorem 1. This theorem generalizes a well-known result of Borsuk (see [2], p. 232).

THEOREM 1. *Let X and Y be two compacta lying in the Hilbert cube Q. If Y is movable and X-like, then $Y \leqslant^q X$.*

Proof. According to a theorem of Spież [13] we can assume that Y is uniformly movable. Then, if V is a closed neighborhood of Y (in Q), there exist a closed neighborhood $V_0 \subset V$ of Y and a shape morphism $\boldsymbol{h}\colon V_0 \to Y$ such that

$$S(i)\cdot \boldsymbol{h} = S(j), \tag{1}$$

where $S(i)$ and $S(j)$ are shape morphisms induced by the inclusions $i\colon Y \to V$ and $j\colon V_0 \to V$. In this situation we say that the neighborhood V_0 is *associated* with V.

By a small modification of the proof of Borsuk's Theorem 12.2 ([2], p. 232) it is easy to see that there are maps $g\colon Y \to X$ and $f\colon X \to V_0$ satisfying the relation

$$f\cdot g \simeq i_0, \tag{2}$$

where $i_0\colon Y \to V_0$ is the inclusion. Then, for

$$S(g)\colon Y \to X \quad \text{and} \quad \boldsymbol{h}\cdot S(f)\colon X \to Y$$

we get shape morphisms that, by (1) and (2), satisfy the relation

$$S(i)\cdot \boldsymbol{h}\cdot S(f)\cdot S(g) = S(j)\cdot S(i_0) = S(i).$$

Thus the proof of Theorem 1 is complete.

Remark 1. As follows from [2], p. 233, the relation $Sh(Y) \leqslant Sh(X)$ cannot be deduced from the hypothesis of Theorem 1. In fact, Borsuk [2] finds two quasi-homeomorphic movable compacta X and Y such that Y is not shape dominated by X.

COROLLARY 1. *Let X and Y be two compacta. If Y is movable and X-like and X is FAR, then Y is FAR.*

Proof. By Theorem 1 we have $Y \leqslant^q X$, and it is an elementary fact that the property FAR is an invariant of quasi-domination (see [6], Corollary 3.3, and [8], Theorem 1).

In the following result we express a simple characterization of quasi-domination for movable compacta.

THEOREM 2. *Let X and Y be movable compacta. Then $Y \leqslant^q X$ if and only if for every neighborhood V of Y (in Q) there exists a map $g\colon X \to V$ such that for some continuous extension $\hat{g}\colon \hat{U} \to V$ of g to a closed neighborhood of X (in Q), the following statement holds*:

($*$) *For every neighborhood $U \subset \hat{U}$ of X (in Q) there exists a map $f\colon Y \to U$ such that $\hat{g} \cdot f \simeq i$, where $i\colon Y \to V$ is the inclusion.*

Proof. Let $\hat{V}$ be a closed neighborhood of Y in Q and V the neighborhood associated with $\hat{V}$ by the uniform movability of Y. Take $\hat{g}\colon \hat{U} \to V$ such that statement ($*$) holds and U is associated with $\hat{U}$. Then we have inclusions

$$i\colon Y \to V, \quad \hat{i}\colon Y \to \hat{V}, \quad j\colon V \to \hat{V},$$
$$\hat{i}_0\colon X \to \hat{U}, \quad j_0\colon U \to \hat{U}$$

and shape morphisms

$$\boldsymbol{h}\colon V \to Y \quad \text{and} \quad \boldsymbol{h}_0\colon U \to X$$

such that

$$S(\hat{i}) \cdot \boldsymbol{h} = S(j) \quad \text{and} \quad S(\hat{i}_0) \cdot \boldsymbol{h}_0 = S(j_0). \tag{3}$$

Now, let $f\colon Y \to U$ be a map satisfying the relation

$$\hat{g} \cdot j_0 \cdot f \simeq i. \tag{4}$$

Setting

$$\boldsymbol{h}_0 \cdot S(f)\colon Y \to X \quad \text{and} \quad \boldsymbol{h} \cdot S(\hat{g}) \cdot S(\hat{i}_0)\colon X \to Y$$

we get shape morphisms that, in virtue of (3) and (4), satisfy the relations

$$\begin{aligned} S(\hat{i}) \cdot \boldsymbol{h} \cdot S(\hat{g}) \cdot S(\hat{i}_0) \cdot \boldsymbol{h}_0 \cdot S(f) &= S(\hat{i}) \cdot \boldsymbol{h} \cdot S(\hat{g}) \cdot S(j_0) \cdot S(f) \\ &= S(\hat{i}) \cdot \boldsymbol{h} \cdot S(i) = S(j) \cdot S(i) = S(\hat{i}). \end{aligned}$$

This proves the part "if" of the theorem. The converse follows immediately.

Remark 2. Theorem 2 allows us to replace shape morphisms by maps in the definition of quasi-domination for movable compacta. In fact, in statement ($*$) we only need the weaker requirement of the existence of $f\colon Y \to U$ for U associated with $\hat{U}$ by the uniform movability of X.

Borsuk proves in [3] that for $Y \in \mathrm{ANR}$ it is equivalent to be quasi-dominated and to be shape dominated by X. In the sequel we prove a more general result.

THEOREM 3. *If Y is an FANR-compactum and $Y \leqslant^q X$, then $Sh(Y) \leqslant Sh(X)$.*

Proof. We can assume that X and Y lie in the Hilbert cube Q. Since Y is FANR, there exists a shape retraction $\boldsymbol{r}\colon V \to Y$, where V is a closed neighborhood of Y (in Q).

Consider now two shape morphisms $\boldsymbol{f}\colon X \to Y$ and $\boldsymbol{g}\colon Y \to X$ such that

$$S(i)\cdot \boldsymbol{f}\cdot \boldsymbol{g} = S(i), \tag{5}$$

where $i\colon Y \to V$ is the inclusion. Setting

$$\boldsymbol{g}\colon Y \to X \quad \text{and} \quad \boldsymbol{r}\cdot S(i)\cdot \boldsymbol{f}\colon X \to Y$$

we get two shape morphisms such that the relation

$$\boldsymbol{r}\cdot S(i)\cdot \boldsymbol{f}\cdot \boldsymbol{g} = \boldsymbol{r}\cdot S(i) = 1_Y$$

(1_Y is the identity morphism) holds in virtue of (5). This proves the theorem.

COROLLARY 2. *Let X and Y be compacta. If Y is FANR and X-like, then* $Sh(Y) \leqslant Sh(X)$.

Proof. It is well known that if Y is FANR, then Y is movable. The corollary follows from Theorems 1 and 3.

3. Some other properties of quasi-domination. In this section we state without proof the following three theorems:

THEOREM 4. *Let $\mathscr{R}$ and $\mathscr{R}'$ be two families of compacta. If for every $A \in \mathscr{R}$ there exists $B \in \mathscr{R}'$ such that $A \leqslant^q B$, then $\mathscr{R}$ is M-dominated by $\mathscr{R}'$.*

(See [11] for definitions of $\mathscr{R}$-movability and M-domination.)

THEOREM 5. *If the compactum Y is quasi-dominated by the compactum X, then the shape coefficients of Lusternik–Schnirelmann of X and Y fulfil the relation $\chi(Y) \leqslant \chi(X)$.*

The definition of quasi-domination can be transferred to pointed spaces in a natural way. The following theorem holds:

THEOREM 6. *If the pointed compactum (Y, y_0) is quasi-dominated by (X, x_0), and (X, x_0) is approximatively n-connected, then (Y, y_0) is approximatively n-connected.*

The proofs of Theorems 4–6 can be obtained with a few changes in the respective theorems for shape domination due to Olędzki ([11], Theorem (2.5)) and Borsuk ([5], Theorem (4.1), and [2], Theorem (8.1), p. 144).

Added in proof. Theorem 3 can be obtained as a consequence of Lemma 3.7 and Theorem 3.10 of L. Boxer and R. B. Sher, *Borsuk's fundamental metric and shape domination*, Bulletin de l'Académie Polonaise des Sciences 26 (1978), p. 849–853. Boxer has recently proved (*Remarks on quasi-domination*, ibidem 30 (1982), p. 553–558) that this result holds for the class of calm compacta introduced by Z. Čerin.

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