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ON A TOPOLOGY OF CONVERGENCE

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1. Let S be an arbitrary set and let X be the set of all sequences $x = \{s_n\}$ such that $s_n \in S$. By S^* we shall denote the collection of all subsets of S.

A function G from X to S^* is called a *G-convergence function* or, shortly, a *G-convergence*. A sequence $x \in X$ is said to be *G-convergent* iff $G(x)$ is a non-empty set. An element $s \in S$ is called a *G-limit* of x iff $s \in G(x)$.

The definition of a G-convergence function is due to Mikusiński.

There is a one-to-one correspondence between the collection of all G-convergence functions and the collection of all classes of convergences. A class of convergence is any set of pairs (x, s), where $x \in X$ and $s \in S$. We shall say that *G-convergence corresponds to $\mathscr{C}$-class of convergence* provided that, for every $x \in X$, $s \in G(x)$ iff $(x, s) \in \mathscr{C}$ and $G(x)$ is the empty set if $(x, s) \notin \mathscr{C}$ for any $s \in S$.

2. In this section we shall consider some types of G-convergence. If $x = \{s_n\}$ is a constant sequence such that $s_n = s$ for each n, then we shall write $G(s)$ instead of $G(x)$ or $G(\{s_n\})$. We say that a subset $A \subset S$ is *G-closed* iff we have $G(x) \subset A$ for any sequence $\{s_n\} = x$ such that $s_n \in A$.

We say that *G-convergence is of type $\mathscr{L}$* iff it satisfies the following conditions:

1° $s \in G(s)$ for each $s \in S$,

2° if y is a subsequence of x, then $G(x) \subset G(y)$,

3° the set $G(x)$ is G-closed for every $x \in X$,

4° if $t_n \in G(s_n)$, then $G(\{t_n\}) \subset G(\{s_n\})$.

G-convergence is said to be *of type $\mathscr{L}^*$* iff it satisfies conditions 1°-4° and, moreover, the following condition:

5° if $s \in G(x)$, then there is a subsequence y of x such that for any subsequence z of y we have $s \in G(z)$.

If $\mathscr{C}$ is a class of convergence of type $\mathscr{L}$, i.e.,

1′ $(x, s) \in \mathscr{C}$ and $(x, t) \in \mathscr{C}$ implies $s \parallel t$, 2′ $(s, s) \in \mathscr{C}$ for each s, where the first s is understood as a sequence whose all terms are equal to s,

3′ $(x, s) \in \mathscr{C}$ implies $(y, s) \in \mathscr{C}$ for every subsequence y of x,

then G-convergence which corresponds to $\mathscr{C}$ is of type $\mathscr{L}$.

Moreover, if $\mathscr{C}$ is of type $\mathscr{L}^*$, i.e., if $\mathscr{C}$ satisfies conditions 1′-3′ and the condition

4′ if $(x, s) \notin \mathscr{C}$, then there exists a subsequence y of x such that we have $(z, s) \notin \mathscr{C}$ for any subsequence z of y,

then G-convergence corresponding to $\mathscr{C}$ is of type $\mathscr{L}^*$.

In fact, if G-convergence corresponds to $\mathscr{C}$ in the sense of section 1, then it satisfies conditions 1°, 2° and 5° by virtue of conditions 2′, 3′ and 4′, respectively. By 1′, we have uniqueness of G-limit of x, i.e., if $s \in G(x)$ and $t \in G(x)$, then $s = t$. Hence, by 1°, we obtain 3° and 4°.

Let F be any family of subsets of S. An element $s \in S$ is said to be *F-limit* of a sequence $x = \{s_n\}$ iff for any set $U \in F$ such that $s \in U$ the sequence x is eventually in U, i.e., there exists an N such that $n \geqslant N$ implies $s_n \in U$.

By a *$C(F)$-convergence* we shall mean a G-convergence such that, for every sequence x, $G(x)$ is the set of all F-limits of x.

The definition of a C-operation is due to Mikusiński.

THEOREM 1. *For any family F of subsets of S, $C(F)$-convergence is of type $\mathscr{L}^*$.*

Proof. It is clear that $s \in C(F)(s)$, which proves 1°. If y is a subsequence of x and s is an F-limit of x, then s is an F-limit of y. Thus $C(F)(y) \supset C(F)(x)$, which proves 2°. Assume that $t_n \in C(F)(x)$ and $s \in C(F)(\{t_n\})$. Let U be an arbitrary element of F such that $s \in U$. There exists an element $t_n \in U$. Since $t_n \in C(F)(x)$ and $t_n \in U$, the sequence is eventually in U by virtue of the definition of $C(F)(x)$. Thus $s \in C(F)(x)$ and, subsequently, $C(F)(\{t_n\}) \subset C(F)(x)$ which implies 3°. Suppose that $t_n \in C(F)(s_n)$ and $s \in C(F)(\{t_n\})$. Then for any $U \in F$ such that $s \in U$ we have $t_n \in U$ for sufficiently large n. Since $t_n \in C(F)(s_n)$, $t_n \in U$ implies $s_n \notin U$ for sufficiently large n. Hence $s \in C(F)(\{s_n\})$, which implies 4°. If an element $s \in S$ is not an F-limit of x, then there is a $U \in F$ and a subsequence $y = \{t_n\}$ of x such that $s \in U$ and $t_n \in U$ for $n = 1, 2, \ldots$ Evidently, $s \in C(F)(z)$ for any subsequence z of y. This proves condition 5°. Thus the proof is complete.

3. Suppose that a G-convergence is of type $\mathscr{L}^*$. There arises the question whether there exists a family F of subsets S such that $C(F) = G$. In the case when G-convergence corresponds to a class $\mathscr{C}$ of convergence of type $\mathscr{L}^*$, the positive answer follows from the results of paper [5] and monograph [4], and then F is a topology for S. In this section we shall give a generalization of these results.

Let G be a convergence from X to S^*. A subset U of S is said to be *G-open* iff, for any sequence $x \in X$, x is eventually in U provided the intersection $U \cap G(x)$ is not empty. By $T(G)$ we shall denote the family of all G-open sets. Evidently, if U is a G-open set, then the complementary $S \setminus U$ is G-closed.

For any G-convergence function, the family $T(G)$ is a topology for S, i.e., the empty set $\emptyset$ and S itself belong to $T(G)$, the union of any number of members of $T(G)$ is again a member of $T(G)$, and the intersection of a finite number of members of $T(G)$ is again a member of $T(G)$.

In the sequel the fact that y is a subsequence of x will be denoted by $y \prec x$.

THEOREM 2. *Suppose that a G-convergence function is of type $\mathscr{L}^*$ and satisfies the following condition:*

6° *for any sequence x there esists a subsequence y such that the set*

$$A = \bigcup_{z \prec y} G(z)$$

is G-closed.

Then, if an element $s \in S$ is a $T(G)$-limit of x, there exists for any subsequence y of x a subsequence z such that $s \in G(z)$.

Proof. Let x be an arbitrary sequence in X and let s be an arbitrary element of S. Suppose that there exists a subsequence y of x such that for any subsequence z of y we have $s \notin G(z)$. We assert that the element s cannot be a $T(G)$-limit of x. In fact, by condition 6°, there exists a subsequence u of z such that the set $A = \bigcup_{t \prec z} G(t)$ is G-closed. Assume that $u = \{s_n\}$ and let $B_k = \bigcup_{n=k}^{\infty} G(s_n)$, where $G(s_n)$ is understood to be the value of a G-convergence function at the constant sequence whose all terms equal s_n.

We shall prove that there exists a k such that $s \notin A \cup B_k$ and the set $A \cup B_k$ is G-closed.

First we shall show that the set $A \cup B_k$ is G-closed for each k. In fact, let $\{p_n\}$ be an arbitrary sequence such that $p_n \notin B_k \cup A$. There exists a subsequence $\{q_n\}$ of $\{p_n\}$ such that $q_n \in A$ or $q_n \in B_k$ for all n. In the first case we have $G(\{p_n\}) \subset G(\{q_n\}) \subset A$, by virtue of postulate 2° and the fact that the set A is G-closed. In the second case there exists a subsequence $\{r_n\}$ of $\{q_n\}$ such that $r_n \in G(s_i)$ for each n and an $i \geqslant k$ or $r_n \in G(s_{i_n})$ with $i_n \to \infty$ as $n \to \infty$. In the first case we have $G(\{p_n\}) \subset G(\{r_n\}) \subset G(s_{i_n}) \subset B_k \cup A$ by virtue of postulates 2° and 4° and the definition of B_k. In the second case we have $G(\{p_n\}) \subset G(\{r_n\}) \subset G(\{s_{i_n}\}) \subset A \cup B_k$ in view of postulates 2° and 4° and the definition of A. Thus we have proved that the set $A \cup B_k$ is G-closed. Now we shall show that there exists a k such that $s \in A \cup B_k$. Since $s \notin G(z)$ for any subsequence z of y, we have $s \notin A$. Suppose that $s \in B_k$ for any k. Then there exists a subsequence j_n of the sequence of integers such that $s \in G(s_{j_n})$ for $n = 1, 2, \ldots$ Hence we have $s \in G(s) \subset G(\{s_{j_n}\}) \subset A$ in view of postulates 1° and 4° and the definition of A. A contradiction with $s \in A$. Now we can prove that the element s cannot be a $T(G)$-limit of x. Really, in view of what we have proved so far, we have $s \notin B_k \cup A$ for a k and the set $B_k \cup A$ is G-closed. Hence $s \in V = S \setminus (B_k \cup A)$ and the set V is G-open. Moreover, by postulate 1°, $s_n \in G(s_n) \subset B_k$ for $n = k, k+1, \ldots$ Thus we have $s_n \notin V$ for $n = k, k+1, \ldots$ Therefore the element s is not a $T(G)$-limit of $\{s_n\}$ and, consequently, s is not a $T(G)$-limit of the sequence x, because $\{s_n\}$ is a subsequence of x. Finally, if an element s is a $T(G)$-limit of x, then for any subsequence y of x there exists a subsequence z such that $s \in G(z)$, which was to be proved.

From Theorem 2 it follows

THEOREM 3. *If G-convergence is of type $\mathscr{L}^*$ and satisfies condition* 6°, *then* $C(T(G)) = G$.

In fact, let x be an arbitrary sequence and $s \in G(x)$. If U is a G-open set and $s \in U$, then x is eventually in U. This proves that s is a $T(G)$-limit of x. Hence $G(x) \subset C(T(G))(x)$. If s is a $T(G)$-limit of x, then, by Theorem 2, for any subsequence y of x there exists a subsequence z such that $s \in G(z)$. Hence, by condition 5°, $s \in G(x)$. Thus we have $C(T(G))(x) \subset G(x)$ which implies the assertion.

THEOREM 4. $C(T(C(F))) = C(F)$, *where F is an arbitrary family of subset of S.*

Proof. It is easy to verify that every set $U \in F$ is $C(F)$-open. Hence we have $T(C(F)) \supset F$. Thus if an element $s \in S$ is a $T(C(F))$-limit of a sequence x, then s is an F-limit of x and, consequently,

$$C(T(C(F)))(x) \subset C(F)(x).$$

If $s \in C(F)(x)$, a set U is $C(F)$-open, and $s \in U$, then x is eventually in U by the definition of a $C(F)$-open set. Therefore s is a $T(C(F))$-limit of x. Hence

$$C(T(C(F)))(x) \supset C(F)(x)$$

and, consequently,

$$C(T(C(F)))(x) = C(F)(x)$$

which implies our assertion.

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