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SOME REMARKS ON THE STEINER TRIPLE SYSTEMS ASSOCIATED WITH STEINER QUADRUPLE SYSTEMS

BY

CHARLES C. LINDNER (AUBURN, ALABAMA)

1. Introduction. A *Steiner quadruple system* (or, more simply, a *quadruple system*) is a pair (Q, q), where Q is a finite set and q is a collection of 4-element subsets of Q (called *blocks*) such that any three distinct elements of Q belong to exactly one block of q. The number $|Q|$ is called the *order of the quadruple system* (Q, q). Hanani [3] proved in 1960 that the spectrum for quadruple systems consisted of the set of all positive integers $n \equiv 2$ or 4 (mod 6). If (Q, q) is a quadruple system and x is any element in Q, we denote $Q \setminus \{x\}$ by Q_x and the set of all triples $\{a, b, c\}$ such that $\{x, a, b, c\} \in q$ by $q(x)$. It is a routine matter to see that $(Q_x, q(x))$ is a Steiner triple system. Two very interesting problems concerning Steiner quadruple systems are the following:

(1) The construction of quadruple systems (Q, q) such that, for some subset X of Q containing *at least two* elements, the Steiner triple systems $(Q_x, q(x))$ and $(Q_y, q(y))$ are non-isomorphic whenever $x \neq y \in X$. If $|X| = n \geqslant 2$ we will say that (Q, q) *has at least n non-isomorphic associated triple systems* (NATS).

(2) The construction of a pair of *non-isomorphic* quadruple systems with the property that the associated triple systems can be isomorphically paired.

In [7], Mendelsohn and Hung have shown that there are exactly four non-isomorphic quadruple systems of order 14. Two of these systems have 2 NATS and the other two are non-isomorphic while having all of their associated triple systems isomorphic to the same Steiner triple system of order 13. Hence there are quadruple systems with at least 2 NATS and there are non-isomorphic quadruple systems whose associated triple systems can be isomorphically paired. As far as the author can tell four quadruple systems of order 14 constructed by Mendelsohn and Hung are the only known systems having property described in (1) or (2). Quadruple systems having property described in (2) are of particular interest since they illustrate the fact that (to within isomorphic) quadruple systems are not necessarily determined by their associated triple systems. The purpose of this paper * is to give a very simple construction for quadruple systems having at least 2 NATS and for pairs of non-isomorphic quadruple systems whose associated triple systems can be isomorphically paired. For a more detailed account of the techniques used in what follows the reader is referred to [4]-[6] and [8].

2. Steiner 3-skeins. By a *3-skein* is meant a pair $(Q, (,,))$, where Q is a finite set and $(,,)$ is a ternary operation on Q such that if in the equation $(x, y, z) = w$ any three elements of x, y, z and w are given, then the remaining element is uniquely determined [1]. It is well known that a quadruple system (Q, q) is equivalent to a 3-skein $(Q, (,,))$ satisfying the following three identities (see, for example, [2]):

$$(x, y, z) = (y, x, z) = (z, y, x), \quad (x, x, y) = y \quad \text{and} \quad \big(x, y, (x, y, z)\big) = z.$$

Such a 3-skein is called a *Steiner* 3-*skein.* In what follows it is convenient to consider quadruple systems algebraically. Although we will use mostly 3-skein and quasigroup terminology we will switch to quadruple triple system vernacular when it facilitates whatever is under discussion. Hence, $(Q, (,,))$ being a Steiner 3-skein and x any element in Q, the idempotent quasigroup $(Q_x, o(x))$ defined for $a \neq b$ by $a\, o(x)\, b = c$ if and only if $(a, b, c) = x$ is a Steiner quasigroup and is, of course, equivalent to the Steiner triple system $(Q_x, q(x))$.

3. Construction of Steiner 3-skeins having at least 2 NATS. Let $(Q, q(,,))$ and $(V, v(,,))$ be any two Steiner 3-skeins. We will set $S = Q \times V$ and denote the direct product of $(Q, q(,,))$ and $(V, v(,,))$ by $(S, qv(,,))$. In what follows we will assume that $Q = \{1, 2, \dots, q\}$ and $V = \{1, 2, \dots, v\}$. If we set $1^* = (1, 1)$, then $(S_{1^*}, o(1^*))$ is the Steiner quasigroup obtained from $(S, qv(,,))$ by deleting $1^* = (1, 1)$. That is to say,

$$(s, x)\, o(1^*)(t, y) = qv\big((1, 1), (s, x), (t, y)\big) = \big(g(1, s, t), v(1, x, y)\big).$$

Now let $(T, o(1^*))$ be any subquasigroup of $(S_{1^*}, o(1^*))$, let $V' = \{v \in V \mid (q, v) \in T\}$ and write $T_v = \{q \in Q \mid (q, v) \in T\}$.

THEOREM 1. *If $(S_{1^*}, o(1^*))$ and $(T, o(1^*))$ and V' are as above, then $|T_v| = |T_w|$ for all $v, w \in V'$. If $T_1 \neq \emptyset$, then $|T_1| = |T_v| - 1$.*

Proof. Let $x \neq y \in V_1'$ and let (s, x) be any element in T_x, and (t, y) any element in T_y. Since $(T, o(1^*))$ is a subquasigroup of $(S_{1^*}, o(1^*))$,

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there is an element $(u, z) \in T$, $z \neq x$ or y, such that $(s, x) o(1^*)(u, z) = (t, y)$. Hence $q(1, s, u) = t$ and $v(1, x, z) = y$. Now let (s', x) be any element in T_x. Since $(u, z) \in T$, we must have

$$(s', x)\, o(1^*)\, (u, z) = \big(q(1, s', u), v(1, x, z)\big) \in T.$$

But $v(1, x, z) = y$ and so $\big(q(1, s', u), y\big) \in T_y$. Hence, if $s' \neq s$, we must have $q(1, s', u) \neq q(1, s, u)$ and it follows that $|T_x| \leqslant |T_y|$. Similarly, $|T_y| \leqslant |T_x|$, so that $|T_x| = |T_y|$.

If $T_1 \neq \emptyset$, let $(s, 1) \in T_1$, $(t, y) \in T_y$, and (u, z) the unique element in T such that $(s, 1)\, o(1^*)\, (u, z) = (t, y)$. Since $v(1, z, y) = y$ we must have $z = y$. Now let $(u, y), (u_1, y), (u_2, y), \ldots, (u_k, y)$ be a distinct listing of the elements in T_y. Then each of $(u, y)\, o(1^*)\, (u_i, y) = \big(q(1, u, u_i), 1\big)$ belongs to T_1, since $|T_1| \geqslant k = |T_y| - 1$ if $u_i \neq u_j$, $q(1, u, u_i) \neq q(1, u, u_j)$. Since for any $(x, 1) \in T_1$ the equation $(u, y)\, o(1^*)\, (a, y) = (x, 1)$ must have a unique solution $(a, y) \in T_y$, it follows that $|T_1| = |T_y| - 1$.

Remark. Theorem 1 is of course true regardless of the element deleted from $S = Q \times V$. We will need this observation in what follows.

THEOREM 2. *Let* $\big(S_{1*}, o(1^*)\big)$, $\big(T, o(1^*)\big)$ *and* V' *be as in Theorem* 1. *Then* $\big(V_1', o(1)\big)$ *is a subquasigroup of* $\big(V_1, o(1)\big)$.

Proof. We need only show closure. So, let $x \neq y \in V_1'$ and let $(s, x), (t, y)$ be any two elements in T. Then

$$(s, x)\, o(1^*)\, (t, y) = \big(q(1, s, t), v(1, x, y)\big) \in T.$$

Since $x \neq y$ and both are different from 1 we must have $v(1, x, y) \in V_1'$. But $x\, o(1)\, y = v(1, x, y)$ and the proof is complete.

Remark. V_1' can of course be the empty set. As with Theorem 1 this last result is true regardless of the point deleted from $S = Q \times V$.

COROLLARY 3. *If* $\big(S_{1*}, o(1^*)\big)$, $\big(T, o(1^*)\big)$, *and* V' *are as in Theorem* 2, *then* $|V'| \equiv 1, 2, 3,$ *or* 4 (mod 6). $|V'| \equiv 1$ *or* 3 (mod 6) *if and only if* $V_1' = V'$.

Now set $\bar{Q} = Q_1 \times \{1\}$. Then $\big(\bar{Q}, o(1^*)\big)$ is as ubquasigroup of $\big(S_{1*}, o(1^*)\big)$ which is isomorphic to $\big(Q_1, o(1)\big)$.

THEOREM 4. *Suppose that* $|Q_1| = 2p - 1 > |V|$, *where* 2 *and* p *are relatively prime, and that* $|Q_1|$ *is not divisible by any* $n \equiv 1$ *or* 3 (mod 6) *such that* $1 < n < |V|$. *Then* $\big(\bar{Q}, o(1^*)\big)$ *is the only subquasigroup of* $\big(S_{1*}, o(1^*)\big)$ *of order* $|Q_1|$.

Proof. Let $\big(T, o(1^*)\big)$ be a subquasigroup of $\big(S_{1*}, (1^*)\big)$ of order $|Q_1|$ and let $V' = \{v \in V \mid (p, v) \in T\}$. If $|V'| = 1$, then $T = \bar{Q}$ and we are through. To see this, suppose $V' = \{x\}$ and let $(s, x), (t, x) \in \big(T, o(1^*)\big)$. Then $(s, x)\, o(1^*)\, (t, x) = \big(q(1, s, t), v(1, x, x)\big) = \big(q(1, s, t), x\big)$. Hence $x = v(1, x, x) = 1$. If $|V'| > 1$, it follows from Theorem 1 that $|T_v| = |T_w| = t \geqslant 2$ for all $v, w \in V_1'$. There are two cases to consider.

(i) $1 \in V'$. In this case $|Q_1| = mt-1$, $|Q| = mt$, where $m = |V'|$. Since $1 \in V'$, $m \equiv 2$ or 4 (mod 6). Since $(T, o(1^*))$ and $(\bar{Q}, o(1^*))$ are subquasigroups, $t-1 = |T_1| = |T \cap \bar{Q}| \equiv 1$ or 3 (mod 6) gives $t \equiv 2$ or 4 (mod 6). Since $|Q|$ is divisible by 2 but not by 4 and $m \geqslant 2$, $t = 0$ which cannot be.

(ii) $1 \notin V'$. In this case $m \equiv 1$ or 3 (mod 6) and so $|Q_1| = mt$, where $1 < m < |V|$. But $|Q_1|$ is not divisible by any integer $\equiv 1$ or 3 (mod 6) strictly between 1 and $|V|$.

Combining (i) and (ii) shows that $|V'| = 1$, which completes the proof.

THEOREM 5. *Let $(Q, q(,,))$ and $(V, v(,,))$ be Steiner 3-skeins with $|Q| = 2p > |V|+1$, 2 and p relatively prime, and $|Q|-1$ not divisible by any $n \equiv 1$ or 3 (mod 6) such that $1 < n < |V|$. Then if $(Q, q(,,))$ has m NATS, then so does the direct product $(Q \times V, qv(,,))$.*

Proof. Suppose that $(Q_1, o(1)), (Q_2, o(2)), \ldots, (Q_m, o(m))$ are NATS of $(Q, q(,,))$. Then, by Theorem 4, $(S_{(1,1)}, o(1, 1)), (S_{(2,1)}, o(2, 1)), \ldots, (S_{(m,1)}, o(m, 1))$ are NATS of $(S = Q \times V, vq(,,))$.

Example. In [7], Mendelson and Hung have constructed a quadruple system (Q, q) of order 14 having 2 NATS. Since 13 is a prime, taking (V, v) to be a quadruple system of order 2, 4, 8, or 10 gives quadruple systems of orders 28, 56, 112, and 140 each having at least 2 NATS. Many more examples along these lines are easily obtained.

4. Non-isomorphic Steiner 3-skeins with isomorphically paired associated Steiner quasigroups. The main results in this section are based on the following theorem, the proof of which can be found in [6]:

THEOREM 6. *Let $(Q, q(,,))$ and $(V, v(,,))$ be Steiner 3-skeins with $(V, v(,,))$ containing no subsystem of order $|Q|$. If for any $n > 1$, where n is the order of a subsystem of $(V, v(,,))$, $|Q|/n \not\equiv 2$ or 4 (mod 6), then the only subsystems of $(Q \times V, qv(,,))$ of order $|Q|$ are the $|V|$ disjoint copies $(Q \times \{v\}, qv(,,))$ of $(Q, q(,,))$ for each v in V.*

COROLLARY 7. *Let $(Q, q_1(,,))$ and $(Q, q_2(,,))$ be non-isomorphic Steiner 3-skeins with isomorphically paired associated Steiner quasigroups. Let $(V, v(,,))$ be a Steiner 3-skein containing no subsystem of order $|Q|$ and such that if $n > 1$ and is the order of a subsystem of $(V, v(,,))$, then $|Q|/n \not\equiv 2$ or 4 (mod 6). Then the direct products $(Q \times V, q_1 v(,,))$ and $(Q \times V, q_2 v(,,))$ are non-isomorphic to isomorphically paired associated Steiner quasigroups.*

Proof. $(Q \times V, q_1 v(,,))$ and $(Q \times V, q_2 v(,,))$ are clearly non-isomorphic. Since the associated Steiner quasigroups in $(Q, q_1(,,))$ and $(Q, q_2(,,))$ can be isomorphically paired, there exist two distinct listings of the elements in Q, $x_1, x_2, \ldots, x_q$ and $y_1, y_2, \ldots, y_q$, such that $(Q_{x_i}, o(x_i))$ and $(Q_{y_i}, o(y_i))$ are isomorphic. Let a_i be any isomorphism from $(Q_{x_i}, o(x_i))$ to $(Q_{y_i}, o(y_i))$. We extend this to a mapping $\bar{a}_i$ from Q onto Q by defining $x_i \bar{a}_i = y_i$.

Now (x_i, j) and (y_i, j) with $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, v$ are two distinct listings of the elements of $Q \times V$.

Claim. The Steiner quasigroup associated with (x_i, j) in $(Q \times V, q_1 v(,,))$ is isomorphic to the Steiner quasigroup associated with (y_i, j) in $(Q \times V, q_2 v(,,))$. Let γ be a map from $Q \times V \setminus \{(x_i, j)\}$ into $Q \times V \setminus \{(y_i, j)\}$ such that $(q, v)\gamma = (q\bar{a}_i, v)$. γ is clearly 1-1 and onto. To see that γ is in fact an isomorphism let (x_1, y_1) and (x_2, y_2) be any two elements in $Q \times V \setminus \{(x_i, j)\}$. Then

$$\begin{aligned}\big((x_1, y_1)\, o(x_i, j)\, (x_2, y_2)\big)\gamma &= \big(q_1(x_i, x_1, x_2), v(j, y_1, y_2)\big)\gamma \\ &= \big(q_1(x_i, x_1, x_2)\bar{a}_i, v(j, y_1, y_2)\big) = \big(q_2(y_i, x_1\bar{a}_i, x_2\bar{a}_i), v(j, y_1, y_2)\big) \\ &= (x_1\bar{a}_i, y_1)\, o(y_i, j)\, (x_2\bar{a}_i, y_2) = (x_1, y_1)\gamma\, o(y_i, j)\, (x_2, y_2)\gamma .\end{aligned}$$

Example. Let (Q, q_1) and (Q, q_2) be the pair of non-isomorphic Steiner quadruple systems of order 14 constructed by Mendelsohn and Hung in [7] with isomorphically paired Steiner triple systems. Actually, all of the associated triple systems are isomorphic to the same Steiner triple system of order 13. Then taking (V, v) to be any quadruple system not containing a subsystem of order 14 gives a pair of non-isomorphic quadruple systems of order $14|V|$ with their associated triple systems isomorphically paired. For example, taking (V, v) to be of order 2, 4, 10, or 20 gives a pair of non-isomorphic quadruple systems of order 28, 56, 140, and 280 with isomorphically paired triple systems.

5. Problems. (1) Construct a Steiner quadruple system of every order $n > 14$ having at least two NATS (**P 934**). (2) Are there any quadruple systems of order n having n NATS? (**P 935**) (3) Construct a pair of non--isomorphic quadruple systems of every order $n \geqslant 14$ having isomorphically paired associated Steiner triple systems. (**P 936**)

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AUBURN UNIVERSITY

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