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ON STEINER MANIFOLDS

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Introduction. The notion of TS-quasigroups is important in the theory of quasigroups developed by R. H. Bruck, A. Sade, S. K. Stein, V. D. Belousov and others (see e.g. [1] and [4]). A quasigroup is called a *TS-quasigroup* if the equation $xy = z$ implies the equation $x'y' = z'$, where x', y', z' is any permutation of elements x, y, z. Put another way, this means that we always have $xy = yx$ and $x(xy) = y$. An idempotent TS-quasigroup is called a *Steiner quasigroup* [1].

The notion of a TS-quasigroup was introduced by Bruck [2] who also pointed out the analogy between the idempotent TS-quasigroups and the triple systems of Steiner. Some results about Steiner quasigroups and their generalizations can be found in [3] and [5]-[7].

In [8], I have considered the notion of an A^k-algebra—a generalization of Steiner quasigroups. Each A^k-algebra is determined by a finite system of equations. The existence of an A^k-algebra of n elements implies the existence of at least one system $\sigma(2, k, n)$ of Steiner that is completely determined by this algebra.

In this paper, I shall generalize the notion of an A^k-algebra and hence also of an idempotent TS-quasigroup, by introducing the notion of a Steiner $(k_1, k_2, \ldots, k_n)$-manifold.

These manifolds appear to correspond precisely to such $\sigma(2, k, m)$-systems of Steiner which can be described algebraically as finitely equationally axiomatizable groupoids.

We get a further generalization by introducing the notion of a Steiner quasimanifold.

1. Let $k_1, k_2, \ldots, k_n$ be an increasing sequence of integers and let $k_1 > 2$.

Definition 1. The $(k_1, k_2, \ldots, k_n)$-*groupoid of Steiner* is the groupoid $G = \langle A, \circ \rangle$ for which the following conditions hold:

(W_1) Each pair of different elements of A generates some subset of A consisting of k_1 elements.

(W_i) ($i = 1, 2, \ldots, n$) Each system of $i+1$ independent (in the sense of generating) elements of A generates some subset of A consisting of k_i elements.

It is easy to see that in the case when A is a finite set, each (k_1)-groupoid of Steiner is equivalent with some $\sigma(2, k, m)$-system of Steiner.

Let $G(X, \circ)$ be the groupoid of words $t_a = t_a(x_0, \ldots, x_n)$, where $x_i \in X$ are free generators. Denote by $T^{(n)}$ the set of all words of $G(X, \circ)$, and by $T^{(i)}$ a subset of $T^{(n)}$ generated by $x_0, x_1, \ldots, x_i$. Let π be a relation of equivalence in $T^{(n)}$ satisfying the following conditions:

(M_j) $|T^{(j)}/\pi| = k_j$ ($j = 1, 2, \ldots, n$);

(RP_0) for each triple of words $t_1, t_2, t_3 \in T^{(n)}$ if $t_1 \pi t_2$, then $(t_3 \circ t_1) \pi (t_3 \circ t_2)$ and $(t_1 \circ t_3) \pi (t_2 \circ t_3)$;

(RP) for each system of $i+1$ words $t_0, t_1, \ldots, t_i \in T^{(n)}$ and for each pair of words $t_a, t_\beta \in T^{(i)}$ if $t_a(x_0, x_1, \ldots, x_i) \pi t_\beta(x_0, x_1, \ldots, x_i)$, then $t_a(t_0, t_1, \ldots, t_i) \pi t_\beta(t_0, t_1, \ldots, t_i)$ for $i = 1, 2, \ldots, n$;

(RJ) for each system of $i+2$ words $t_0, t_1, \ldots, t_{i+1} \in T^{(i)}$ such that $\sim (t_j \pi t_s)$ for $j \neq s$ and $j, s = 0, 1, \ldots, i$ there exists a word $t \in T^{(i)}$ for which $t(t_0, t_1, \ldots, t_i) \pi t_{i+1}$.

Conditions (M_j), (RP_0), (RP) and (RJ) are a scheme of a system of axioms. We get a well defined system $\langle T^{(n)}, \pi \rangle$ of axioms by specifying $T^{(n)}$ and π. By a *Steiner* $(k_1, k_2, \ldots, k_n)$*-manifold* we shall mean the class of algebras satisfying a system $\langle T^{(n)}, \pi \rangle$ of axioms.

We call a finite axiom system of a Steiner $(k_1, k_2, \ldots, k_n)$-manifold a *submodel* $\langle \tau, \pi \rangle$ of $\langle T^{(n)}, \pi \rangle$ if $\tau \subset T^{(n)}$ is finite and the relation π in $T^{(n)}$ is well defined by its own restriction to τ.

2. Let ϱ be a relation of equivalence in $T^{(n)}$ for which

(M_n) $|T^{(n)}/\varrho| = k$

and

(RP_0) for each triplet of words $t_1, t_2, t_3 \in T^{(n)}$ if $t_1 \varrho t_2$, then $(t_3 \circ t_1) \varrho (t_3 \circ t_2)$ and $(t_1 \circ t_3) \varrho (t_2 \circ t_3)$.

Definition 2. The *natural sequence* of subsets $T_0, T_1, \ldots, T_i, \ldots$ of $T^{(n)}$ is the sequence constructed as follows:

1. T_0 consists of all free generators of $T^{(n)}$,

2. T_j consists of all words of the form $t_i \circ t_s$, where either $t_i \in T_{j-1}$ and $t_s \in T_p$ ($p \leqslant j-1$) or $t_s \in T_{j-1}$ and $t_i \in T_p$.

Definition 3. A *basis* of the natural sequence of subsets $T_0, T_1, \ldots$ of $T^{(n)}$ is a sequence of words $\tau_1, \ldots, \tau_k$ such that 1° $\tau_i \varrho \tau_j$, $i \neq j$ ($i, j = 1, 2, \ldots, k$), and 2° if $\tau_i \in T_r$, then for each $t \in T_s$ ($s < r$) we have $\sim t \varrho \tau_i$.

Having chosen a basis, let T_q be the last among those elements of the natural sequence which contains some elements of the basis. Put

$$T = \sum_{p=0}^{q+1} T_p.$$

In virtue of (RP_0) it is easy to prove by induction that the relation ϱ in $T^{(n)}$ is well defined by its own restriction to T. Therefore:

For each axiom system $\langle T^{(n)}, \pi\rangle$ *there exists a finite axiom system* $\langle \tau, \pi\rangle$.

3. Let $\langle T^{(1)}, \pi\rangle$ be an axiom system of a Steiner (k_1)-manifold. Free generators of $T^{(1)}$ will be denoted by x and y.

THEOREM 1. *For the axiom system* $\langle T^{(1)}, \pi\rangle$ *we have* $(x\circ x)\pi x$.

Proof. Suppose that

$$(*) \qquad \sim (x\circ x)\pi x.$$

Moreover, let each coset of π contain an element $t_i(x, x)$. Let the word $t(x, y)$ belong to the coset containing $t_j(x, x)$. By (RP), $t(x, y)\pi t_j(x, x)$ implies $t(x, x)\pi t_j(x, x)$. Hence all words that are equishaped with $t(x, x)$ are in relation π with $t(x, x)$ (words $t_1(x, y)$ and $t_2(x, y)$ are equishaped if $t_1(x, x)$ is identical with $t_2(x, x)$). Since x and y are equishaped, we have $x\pi y$. But by (RP) we have $x\pi t$, where t is an arbitrary word. Hence $x\pi(x\circ x)$ which contradicts $(*)$.

Hence there exists a coset not containing a word which could be written by means of x only. Denote such a class by K. It follows form (RJ) that if K contains the word $t(x, y)$, then there exists a word $t_1(x, y)$ such that $t(x, y)\pi t_1(x, x\circ x)$. But $t_1(x, x\circ x)$ is written by means of x alone — a contradiction. Thus $(x\circ x)\pi x$, which completes the proof.

THEOREM 2. *For the axiomatics* $\langle T^{(1)}, \pi\rangle$ *we have:*

(Q_1) *for each triplet of words* $t_1, t_2, t_3 \in T^{(1)}$ *if* $t_1\circ t_2 \pi t_1\circ t_3$, *then* $t_2\pi t_3$;

(Q_1') *for each triplet of words* $t_1, t_2, t_3 \in T^{(1)}$ *if* $t_2\circ t_1 \pi t_3\circ t_1$, *then* $t_2\pi t_3$;

(Q_2) *for each pair of words* t_1, t_2 *there exists a word* t_3 *such that* $t_1\circ t_3\pi t_2$;

(Q_2') *for each pair of words* t_1, t_2 *there exists a word* t_3 *such that* $t_3\circ t_1\pi t_2$.

Proof. We start with the proof of (Q_1). If $x\pi y$, then, by (RP), $x\pi t$ for each t, which is impossible. If $x\pi x\circ y$, then by (RP) we have $y\pi y\circ x$, whence $k_1 = 2$, a contradiction. In the same way we obtain $\sim x\pi(y\circ x)$. Thus

(i) $\sim x\pi(x\circ y)$, $\sim x\pi(y\circ x)$ *and* $\sim x\pi y$.

Let $t_1\circ t_2\pi t_1\circ t_3$. By (i) and (RJ) there exists a t such that $y\pi t(x, x\circ y)$. Hence by (RP) we have first $t_2\pi t(t_1, t_1\circ t_2)$ and $t_3\pi t(t_1, t_1\circ t_3)$, which by

(RP) and the assumption of the theorem implies $t_3\pi t(t_1, t_1\circ t_2)$, and, finally, $t_2\pi t_3$.

We prove (Q_1') in the same way.

From (Q_1), (Q_1'), (RP) and (M_1) we immediately obtain (Q_2) and (Q_2').

Theorems 1 and 2 lead to the

COROLLARY. *Each Steiner $(k_1, k_2, \ldots, k_n)$-manifold is an idempotent quasigroup.*

4. Example of a construction. Let $t(x_1, x_2, \ldots, x_n)\in T^{(n)}$ and $y_1, y_2, \ldots, y_n$ be coordinates of points of the hyperplane $y_1+y_2+\ldots+y_n = 1$ in the n-dimensional affine space over the field $\mathrm{GF}(p)$. Let h be the mapping $\rightarrow (y_1, y_2, \ldots, y_n)$ defined as follows:

$$h\colon x_i \rightarrow (y_1, y_2, \ldots, y_n), \quad \text{where } y_i = 1,\ y_j = 0 \text{ for } j \neq i.$$

If $h\big((t_1(x_1, x_2, \ldots, x_n)\big) = (y_1', y_2', \ldots, y_n')$ and $h\big(t_2(x_1, x_2, \ldots, x_n)\big) = (y_1'', y_2'', \ldots, y_n'')$, then

$$h\big(t_1(x_1, x_2, \ldots, x_n)\circ t_2(x_1, x_2, \ldots, x_n)\big) = (y_1, y_2, \ldots, y_n),$$

where $y_i = qy_i' + (1-q)y_i''$, q being any element of the field different from 0 and 1.

Define now relation π as follows:

$$t_1(x_1, x_2, \ldots, x_n)\pi t_2(x_1, x_2, \ldots, x_n)$$

if and only if $(y_1', y_2', \ldots, y_n') = (y_1'', y_2'', \ldots, y_n'')$.

The pair $\langle T^{(n)}, \pi\rangle$, thus defined, is an axiom system of a Steiner $(p, p^2, \ldots, p^{n-1})$-manifold. It is easy to verify that each Steiner (p)-manifold obtained in this manner satisfies the condition of elasticity:

$$x\circ(y\circ x) = (x\circ y)\circ x.$$

A Steiner (p, p^2)-manifold is a distributive quasigroup, i.e. it satisfies the following conditions:

$$x\circ(y\circ z) = (x\circ y)\circ(x\circ z),$$

$$(y\circ z)\circ x = (y\circ x)\circ(z\circ x).$$

A Steiner (p, p^2, p^3)-manifold is a medial quasigroup (according to Stein); this means that it satisfies the equation

$$(x\circ y)\circ(u\circ v) = (x\circ u)\circ(y\circ v).$$

5. Examples of axiom systems.

I. $x\circ x = x$.

II. $x\circ y = y\circ x$.

III. $x\circ(x\circ y) = y$.

This is an axiom system of Steiner (3)-manifold. From II the elasticity follows.

IV. $x\circ(y\circ z) = y\circ\big(z\circ(x\circ y)\big)$.

I-IV are axioms of Steiner (3,9)-manifold. It is easy to prove that they imply the distributivity conditions.

V. $x\circ\big(y\circ(z\circ u)\big) = z\circ\big(y\circ(x\circ u)\big)$.

I-V are axioms of Steiner (3, 9, 27)-manifold. They imply the condition of mediality.

6. The review of Steiner (k_1)-manifolds for $k_1 \leqslant 7$. There exists one and only one Steiner (3)-manifold:

[A³] $$x\circ x = x, \quad x\circ y = y\circ x, \quad x\circ(x\circ y) = y.$$

There exists one and only one Steiner (4)-manifold:

[A⁴] $$x\circ x = x, \quad x\circ y = y\circ(y\circ x), \quad x\circ\big(x\circ(x\circ y)\big) = y.$$

There exist three Steiner (5)-manifolds:

[A⁵] $$x\circ x = x, \quad x\circ y = y\circ x, \quad x\circ(x\circ y) = y\circ\big(y\circ(y\circ x)\big),$$
$$x\circ\Big(x\circ\big(x\circ(x\circ y)\big)\Big) = y.$$

[A⁵′] $$x\circ x = x, \quad x\circ y = y\circ(y\circ x), \quad x\circ\big(x\circ(x\circ y)\big) = y\circ\big(y\circ(y\circ x)\big),$$
$$x\circ\Big(x\circ\big(x\circ(x\circ y)\big)\Big) = y.$$

[A⁵″] dual to [A⁵′]-with the operation $x\times y = y\circ x$.

There exist five Steiner (7)-manifolds:

[A⁷] $$x\circ x = x, \quad x\circ y = y\circ\Big(y\circ\Big(y\circ\big(y\circ(y\circ x)\big)\Big)\Big),$$
$$x\circ(x\circ y) = y\circ(y\circ x), \quad x\circ\big(x\circ(x\circ y)\big) = y\circ\Big(y\circ\big(y\circ(y\circ x)\big)\Big),$$
$$x\circ\Big(x\circ\Big(x\circ\Big(x\circ\big(x\circ(x\circ y)\big)\Big)\Big)\Big) = y.$$

[A⁷′] dual to [A⁷].

[B⁷] $$x\circ x = x, \quad x\circ y = y\circ x. \quad \big(x\circ(x\circ y)\big)\circ\big(y\circ(y\circ x)\big) = x\circ y,$$
$$(x\circ y)\circ\big(y\circ(y\circ x)\big) = y\circ\big(x\circ(x\circ y)\big), \quad x\circ\big(x\circ(x\circ y)\big) = y.$$

[C⁷] $$x\circ x = x, \quad x\circ(x\circ y) = y, \quad (y\circ x)\circ x = (x\circ y)\circ y,$$
$$\big((y\circ x)\circ x\big)\circ x = y, \quad (x\circ y)\circ(y\circ x) = y\circ(x\circ y).$$

[C⁷′] dual to [C⁷].

7. Steiner quasimanifolds. Let $G(X, \circ)$ be a groupoid of words, where X is the set of three free generators: $X = \{x, y, z\}$. Denote by $T^{(2)}$ the set of all words of this groupoid. $T^{(1)}$ denotes the subset of words of this groupoid generated by x and y.

Let π be an equivalence relation in $T^{(2)}$ satisfying the conditions

(M_1) $|T^{(1)}/\pi| = k_1$;

(RP_0) for each triplet of words $t_1, t_2, t_3 \in T^{(2)}$ if $t_1 \pi t_2$, then $t_3 \circ t_1 \pi t_3 \circ t_2$ and $t_1 \circ t_3 \pi t_2 \circ t_3$;

(RP_1) for each pair of words $t_0, t_1 \in T^{(2)}$ and for each pair of words $t_\alpha, t_\beta \in T^{(1)}$, if $t_\alpha(x, y) \pi t_\beta(x, y)$, then $t_\alpha(t_0, t_1) \pi t_\beta(t_0, t_1)$;

(RJ_1) for each triplet of words $t_0, t_1, t_2 \in T^{(1)}$ such that $\sim (t_j \pi t_s)$ for $j \neq s$ and $j, s = 0, 1, 2$, there exists a word $t \in T^{(1)}$ such that $t(t_0, t_1) \pi t_2$.

Definition. The words $t_0, t_1, t_2 \in T^{(2)}$ are *linearly dependent* which we denote by $L(t_0, t_1, t_2)$, if $\sim (t_j \pi t_s)$ for $t \neq s$ and $t, s = 0, 1, 2$, and if there exists a word $t \in T^{(1)}$ such that $t(t_0, t_1) \pi t_2$.

Let ϱ be an equivalence relation in the set $T^{(2)}$ such that

(W) $t_1 \pi t_2$ implies $t_1 \varrho t_2$;

(M_2) $|T^{(2)}/\varrho| = k_2$;

(RP_2) for each triplet of words $t_0, t_1, t_2 \in T^{(2)}$ such that $\sim L(t_0, t_1, t_2)$ and for each pair of words $t_\alpha, t_\beta \in T^{(2)}$, if $t_\alpha \varrho t_\beta$, then $t_\alpha(t_0, t_1, t_2) \varrho t_\beta(t_0, t_1, t_2)$;

(RJ_2) for each quadruple of words $t_0, t_1, t_2, t_3 \in T^{(2)}$ such that $\sim t_j \varrho t_s$ for $j \neq s$ and $j, s = 0, 1, 2, 3$, and $\sim L(t_0, t_1, t_2)$, there exists a word $t \in T^{(2)}$ such that $t(t_0, t_1, t_2) \varrho t_3$.

A model $\langle T^{(2)}, \varrho \rangle$ is called *axiomatics* of a (k_1, k_2)-quasimanifold of Steiner.

Each Steiner (k_1, k_2)-manifold is a Steiner (k_1, k_2)-quasimanifold. In fact, conditions (RP_2) and (RJ_2) are a weakened form of conditions (RP) and (RJ) for the relation π of section 1.

We give now an example of an axiom system of a Steiner quasimanifold which is not an axiom system of a Steiner manifold.

(3,7)-quasimanifold of Steiner:

I. $x \circ x = x$,

II. $x \circ y = y \circ x$,

III. $x \circ (x \circ y) = y$,

IV. $\sim L(x, y, z) \to x \circ (y \circ z) = (x \circ y) \circ z$, where $\sim L(x, y, z)$ denotes the independence in the sense of generating.

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