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DIFFERENCE PROPERTIES OF HIGHER ORDERS FOR CONTINUITY AND RIEMANN INTEGRABILITY

BY

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1. Introduction. Throughout this paper, N, Z, and R will always denote the sets of all non-negative integers, integers, and reals, respectively.

The notion of the so-called difference property was first introduced by de Bruijn in [1]. He asked what can be said about a function $f\colon R \to R$ which satisfies the following condition:

(1) for each $h \in R$ the function $\Delta_h f\colon R \to R$ defined by

$$\Delta_h f(x) := f(x+h) - f(x), \qquad x \in R,$$

belongs to a given class $\mathscr{F} \subset R^R$.

It was shown that for a great number of important classes $\mathscr{F}$ the function f may be written in the form

(2) $f = \Gamma + g$, where $g \in \mathscr{F}$ and $\Gamma\colon R \to R$ is an additive function, i.e., it satisfies the Cauchy functional equation

$$\Gamma(x+y) = \Gamma(x) + \Gamma(y), \qquad x, y \in R.$$

If any function $f\colon R \to R$ satisfying (1) is of the form (2), then the class $\mathscr{F}$ is said to have the *difference property*. De Bruijn ([1], [2]) proved the difference property for the class of all continuous functions, k times differentiable functions, analytic functions, absolutely continuous functions, functions with bounded variation, and for some other classes. Afterwards, some of de Bruijn's results have been generalized in various directions (cf., e.g., [3], [4], [6], and [8]). Among others, Carroll [3] pointed out the difference property for the class of all complex-valued continuous functions defined on a locally compact Abelian group and for the class of all complex-valued Riemann integrable functions on a locally compact second countable Abelian group. In the case of compact groups the commutativity assumption may be omitted and some stronger results hold true. On the other hand, Carroll and Koehl ([4], [8]) noted that instead of complex-valued functions one can consider functions with values in a complex Banach space to obtain similar results.

In connection with what was mentioned above the following considerations seem to be very natural. Let $(G, +)$ be a locally compact Abelian group and let $(X, \|\cdot\|)$ be a complex Banach space. Given $h_1, \ldots, h_n \in G$ we define inductively the difference operator $\Delta_{h_1 \ldots h_n}$ with increments $h_1, \ldots, h_n$ as follows:

$$\Delta_{h_1} f(x) := f(x+h_1) - f(x), \qquad x \in G;$$

$$\Delta_{h_1 \ldots h_n} f := \Delta_{h_n}(\Delta_{h_1 \ldots h_{n-1}} f), \qquad f \in X^G.$$

If $h_1 = \ldots = h_n = h$, then we write briefly $\Delta_h^n f$ instead of $\Delta_{\underbrace{h \ldots h}_{n}} f$. Additionally, we put $\Delta_h^0 f = f$.

A function $f\colon G \to X$ is said to be a *polynomial function of n-th order* if and only if

$$\Delta_h^{n+1} f(x) = 0 \qquad \text{for all } h, x \in G.$$

It is well known (cf. [5], Theorem 3) that $f\colon G \to X$ is a polynomial function of n-th order if and only if it has a (unique) representation

$$f = f_0 + f_1 + \ldots + f_n,$$

where f_0 is a constant vector from X, and $f_i\colon G \to X$ for $i = 1, \ldots, n$ are diagonalizations of i-additive symmetric functions $F_i\colon G^i \to X$, i.e.,

$$f_i(x) := F_i(x, \ldots, x), \qquad x \in G, \ i = 1, \ldots, n.$$

Following Kemperman [7] we introduce

Definition. A class $\mathcal{F} \subset X^G$ is said to have the *difference property of n-th order* $(n \in N)$ if and only if any function $f\colon G \to X$ such that

(3) $$\Delta_h^n f \in \mathcal{F} \qquad \text{for each } h \in G$$

admits a decomposition

(4) $f = \Gamma + g$, where $\Gamma\colon G \to X$ is a polynomial function of n-th order and $g \in \mathcal{F}$.

Suppose $\mathcal{F}$ is closed with respect to the addition and let it contain all constant functions. If for each $h \in G$ the operator $\Delta_h^n|_{\mathcal{F}}$ does not lead out of $\mathcal{F}$, then every function f of the form (4) satisfies (3). Indeed, it suffices to note that the operator Δ_h^n is linear and if Γ is a polynomial function of n-th order, then $\Delta_h^n \Gamma$ is constant. Since any polynomial function of first order is a sum of a constant and an additive function, the class $\mathcal{F}$ has the difference property of first order if and only if it has the difference property in de Bruijn's sense.

The natural question arises which of the classes considered by de Bruijn have difference properties of higher orders. The purpose of this paper is to prove that for any $n \in N$ the class of all continuous functions on a locally compact Abelian group and the class of all Riemann integrable functions on a compact second countable Abelian group have the difference property of n-th order.

2. Difference properties on compact groups. In the sequel the phrase "Banach-valued function" will refer to a function with values in a complex Banach space X. Riemann integrability of a function defined on a locally compact Abelian group G will always be understood as Riemann integrability with respect to the Haar measure on G. All topological groups are supposed to be Hausdorff.

We start with some preliminary lemmas.

LEMMA 1. *Let $(G, +)$ be a locally compact Abelian group and let $f \in X^G$. If for each $h \in G$ the function $\Delta_h^n f$ is continuous (Riemann integrable), then so is the function $\Delta_{h_1 \dots h_n} f$ for any system of n elements $h_1, \dots, h_n \in G$.*

One can derive this lemma from Theorem 2 in [5] which, under our assumptions, guarantees that

$$\Delta_{h_1 \dots h_n} f(x) = \sum_{i \in J} r_i \Delta_{u_i}^n f(x + v_i), \qquad x \in G,$$

where J is a finite set, r_i are some rational numbers, and $u_i, v_i \in G$ depend only on $h_1, \dots, h_n$ for $i \in J$.

LEMMA 2. *Suppose $(G, +)$ is an Abelian group. Let $F \colon G^n \to X$ be a symmetric n-additive function and let $f \colon G \to X$ be the diagonalization of F. Then for any $x, k_1, \dots, k_{n-1} \in G$ we have*

$$\Delta_{k_1 \dots k_{n-1}} f(x) = \frac{n!}{2} \sum_{i=1}^{n-1} F(k_1, \dots, k_i, k_i, \dots, k_{n-1}) + \\ + n!\, F(x, k_1, \dots, k_{n-1}). \tag{5}$$

Proof. We shall make use of the following two known formulae for diagonalizations of symmetric n-additive functions (see [5], Lemma 2 and formula (15)):

$$\Delta_{k_1 \dots k_p} f(x) = \begin{cases} n!\, F(k_1, \dots, k_p) & \text{for } p = n, \\ 0 & \text{for } p > n, \end{cases} \tag{6}$$

$$\Delta_k f(x) = \sum_{i=0}^{n-1} \binom{n}{i} F(\underbrace{x, \dots, x}_{i}, \underbrace{k, \dots, k}_{n-1}). \tag{7}$$

The proof of our lemma will be by induction on n. For $n = 1$, relation (5) is trivially fulfilled. Now, assume (5) is true for some $n \geqslant 1$ and let f be the diagonalization of a symmetric $(n+1)$-additive function $F\colon G^{(n+1)} \to X$. Then, according to (6), (7), and the induction hypothesis, we get

$$\begin{aligned}
\Delta_{k_1\ldots k_n} f(x) &= \Delta_{k_2\ldots k_n}(\Delta_{k_1} f(x)) \\
&= \Delta_{k_2\ldots k_n}\left(\sum_{i=0}^{n}\binom{n+1}{i} F(\underbrace{x, \ldots, x}_{i}, \underbrace{k_1, \ldots, k_1}_{n+1-i})\right) \\
&= \Delta_{k_2\ldots k_n}\binom{n+1}{n-1} F(k_1, k_1, \underbrace{x, \ldots, x}_{n-1}) + \Delta_{k_2\ldots k_n}\binom{n+1}{n} F(k_1, \underbrace{x, \ldots, x}_{n}) \\
&= \binom{n+1}{n-1}(n-1)!\, F(k_1, k_1, k_2, \ldots, k_n) + \\
&\quad + \binom{n+1}{n}\frac{n!}{2}\sum_{i=2}^{n} F(k_1, \ldots, k_i, k_i, \ldots, k_n) + \binom{n+1}{n} n!\, F(k_1, \ldots, k_n, x) \\
&= \frac{(n+1)!}{2}\sum_{i=1}^{n} F(k_1, \ldots, k_i, k_i, \ldots, k_n) + (n+1)!\, F(x, k_1, \ldots, k_n).
\end{aligned}$$

Thus we obtain (5) for $n+1$. This completes the proof.

LEMMA 3. *Let $(G, +)$ be a topological locally compact group and let X be a locally convex metrizable linear topological space. If $\psi\colon G \to X$ is a function bounded on some open non-empty set $U \subset G$ and for every $h \in G$ the function $\Delta_h \psi\colon G \to X$ is continuous, then ψ is continuous.*

For the proof of our lemma the following result of Namioka [9] is useful:

LEMMA 4 (cf. [9], Theorem 4.1). *Let T be a locally compact regular topological space and let X be a locally convex metrizable linear topological space. If $\psi\colon T \to X$ is a weakly continuous function, then there exists a dense G_δ-set $A \subset T$ such that ψ is continuous at each point of A.*

Proof of Lemma 3. Let us choose an arbitrary continuous functional $x^* \in X^*$. There is a neighbourhood V of zero in X such that x^* is bounded on V. Since $\psi(U) \subset \alpha V$ for some $\alpha > 0$, the function $x^* \circ \psi$ is bounded on U. Moreover, for each $h \in G$ the function

$$\Delta_h(x^* \circ \psi) = x^* \circ \Delta_h \psi$$

is continuous. By virtue of Lemma 2.2 in [3], $x^* \circ \psi$ is continuous, and since x^* has been arbitrarily chosen, ψ is weakly continuous. Lemma 4 implies, in particular, that ψ is continuous at some point $x_0 \in G$. The equality

$$\psi(x) = \Delta_{-x_0+x_1}\psi(x - x_1 + x_0) + \psi(x - x_1 + x_0), \qquad x \in G,$$

ensures the continuity of ψ at any point $x_1 \in G$, which was to be shown.

It is worth-while to note that Lemma 3 enables one to apply (without any essential changes) the methods used in [3] in order to prove the following

THEOREM 1. *Let* $(G, +)$ *be a locally compact topological group (not necessarily Abelian) and suppose* X *is a Banach space. If* $f\colon G \to X$ *is such that for each* $h \in G$ *the function*

$$G \ni x \to \Delta_h f(x) := f(x+h) - f(x)$$

is continuous and the function

$$G \ni x \to \nabla_h f(x) := f(h+x) - f(x)$$

is Borel measurable, then $f = \Gamma + g$, *where* $g\colon G \to X$ *is continuous and* $\Gamma\colon G \to X$ *is an additive function.*

Carroll and Koehl have proved Theorem 1 under the additional assumption that G is metrizable (cf. [4], Theorem 2.2). Koehl has removed the metrizability assumption but he had to assume that G is Abelian (cf. [8], Theorems 2.1 and 2.2).

We shall also make use of Lemma 3 in the proof of the principal results of this section which read as follows:

THEOREM 2. *Let* $(G, +)$ *be a compact Abelian topological group. Then for each* $n \in N$ *the class of all Banach-valued continuous functions defined on* G *has the difference property of* n*-th order.*

THEOREM 3. *Suppose* $(G, +)$ *is a compact second countable Abelian topological group. Then for each* $n \in N$ *the class of all Banach-valued Riemann integrable functions defined on* G *has the difference property of* n*-th order.*

The proofs of Theorems 2 and 3 are based on similar ideas, and therefore they will proceed paralelly. Suitable changes will be enclosed in parentheses.

Proof of Theorem 2 (and 3). For $n = 0$ both theorems are obvious. Suppose that they hold true for some $n-1 \in N$. Take an $f\colon G \to X$ such that for each $h \in G$ the function $\Delta_h^n f$ is continuous (Riemann integrable). Let μ denote the Haar measure on G with $\mu(G) = 1$.

We define the function $F_n\colon G^n \to X$ by

$$F_n(h_1, \ldots, h_n) := \frac{1}{n!} \int_G \Delta_{h_1 \ldots h_n} f(y)\, d\mu(y), \qquad h_1, \ldots, h_n \in G. \tag{8}$$

The commutativity of superpositions of difference operators implies the symmetry of F_n. For any $h_1, \ldots, h_{n-1}, h_n', h_n'' \in G$ we have

$$\begin{aligned} F_n(h_1, \ldots, h_{n-1}, h_n' + h_n'') &= \frac{1}{n!} \int_G \Delta_{h_1 \ldots h_{n-1}, h_n' + h_n''} f(y)\, d\mu(y) \\ &= \frac{1}{n!} \int_G \left[\Delta_{h_1 \ldots h_{n-1}, h_n'} f(y + h_n'') + \Delta_{h_1 \ldots h_{n-1}, h_n''} f(y)\right] d\mu(y) \end{aligned}$$

$$= \frac{1}{n!} \int_G \Delta_{h_1 \dots h_{n-1}, h'_n} f(y + h''_n)\, d\mu(y) + \frac{1}{n!} \int_G \Delta_{h_1 \dots h_{n-1}, h''_n} f(y)\, d\mu(y)$$

$$= F_n(h_1, \dots, h_{h-1}, h'_n) + F_n(h_1, \dots, h_{n-1}, h''_n).$$

Together with the symmetry of F_n this means that F_n is a symmetric n-additive function.

Let f_n be the diagonalization of F_n and let $g_n := f - f_n$.

Now, fix arbitrarily a system of $n-1$ elements $k_1, \dots, k_{n-1} \in G$ and put

$$\psi(x) := \Delta_{k_1 \dots k_{n-1}} g_n(x), \qquad x \in G.$$

By Lemma 2 and (8) we obtain

$$\psi(x) = \Delta_{k_1 \dots k_{n-1}} f(x) - \Delta_{k_1 \dots k_{n-1}} f_n(x)$$

$$= \Delta_{k_1 \dots k_{n-1}} f(x) - \frac{n!}{2} \sum_{i=1}^{n-1} F_n(k_1, \dots, k_i, k_i, \dots, k_{n-1}) -$$

$$- n!\, F_n(x, k_1, \dots, k_{n-1})$$

$$= \Delta_{k_1 \dots k_{n-1}} f(x) - \frac{1}{2} \sum_{i=1}^{n-1} \int_G \Delta_{k_1 \dots k_i, k_i \dots k_{n-1}} f(y)\, d\mu(y) -$$

$$- \int_G \Delta_{x, k_1 \dots k_{n-1}} f(y)\, d\mu(y)$$

$$= \int_G \Big[\Delta_{k_1 \dots k_{n-1}} f(x) - \frac{1}{2} \sum_{i=1}^{n-1} \Delta_{k_1 \dots k_i, k_i \dots k_{n-1}} f(y) -$$

$$- \Delta_{x, k_1 \dots k_{n-1}} f(y)\Big]\, d\mu(y), \qquad x \in G.$$

Setting

$$\varphi_x(y) := \Delta_{k_1 \dots k_{n-1}} f(x) - \frac{1}{2} \sum_{i=1}^{n-1} \Delta_{k_1 \dots k_i, k_i \dots k_{n-1}} f(y) -$$

$$- \Delta_{x, k_1 \dots k_{n-1}} f(y), \qquad x, y \in G,$$

in view of Lemma 1, we get for each fixed $x \in G$ a continuous (Riemann integrable) function $\varphi_x \colon G \to X$. Moreover,

$$\Delta_h \varphi_x(y) = -\frac{1}{2} \sum_{i=1}^{n-1} \Delta_h \Delta_{k_1 \dots k_i, k_i \dots k_{n-1}} f(y) -$$

$$- [\Delta_{h, k_1 \dots k_{n-1}} f(y + x) - \Delta_{h, k_1 \dots k_{n-1}} f(y)], \qquad h, x, y \in G.$$

Since every continuous function on a compact domain (every Riemann integrable function) is bounded, for each $h \in G$ the values of $\|\Delta_h \varphi_x(y)\|$ are uniformly bounded in all $x, y \in G$. By virtue of Lemma 2.1 in [3] which remains valid for Banach-valued functions if the absolute value sign is replaced by the norm sign, we infer that $\|\varphi_x(y)-\varphi_x(0)\|$ is uniformly bounded in all $x, y \in G$. Let us note that

$$\begin{aligned}\varphi_x(0) &= \Delta_{k_1\dots k_{n-1}} f(x) - \frac{1}{2}\sum_{i=1}^{n-1} \Delta_{k_1\dots k_i,k_i\dots k_{n-1}} f(0) - \\ &\quad - \Delta_{k_1\dots k_{n-1}} f(x) + \Delta_{k_1\dots k_{n-1}} f(0) \\ &= -\frac{1}{2}\sum_{i=1}^{n-1} \Delta_{k_1\dots k_i,k_i\dots k_{n-1}} f(0) + \Delta_{k_1\dots k_{n-1}} f(0),\end{aligned}$$

which is constant as a function of x. Consequently, there exists an $M > 0$ such that $\|\varphi_x(y)\| \leqslant M$ for all $x, y \in G$. Hence

$$\|\psi(x)\| \leqslant \int_G \|\varphi_x(y)\|\, d\mu(y) \leqslant M, \qquad x \in G,$$

and since

$$\begin{aligned}\Delta_h \psi(x) &= \Delta_{h,k_1\dots k_{n-1}} f(x) - \Delta_{h,k_1\dots k_{n-1}} f_n(x) \\ &= \Delta_{h,k_1\dots k_{n-1}} f(x) - n!\, F_n(h, k_1, \dots, k_{n-1}), \qquad x \in G,\end{aligned}$$

in the case where the assumptions of Theorem 2 are satisfied the function $\Delta_h \psi$ is continuous for each $h \in G$. From Lemma 3 it follows that ψ is continuous.

Under the assumptions of Theorem 3 the function $\varphi\colon G^2 \to X$ determined by

$$\varphi(x, y) := \varphi_x(y), \qquad (x, y) \in G^2,$$

fulfils all the conditions assumed in Lemma 2.3 of [3] with $X = Y = G$. It is enough to check that, for any fixed $y \in G$, $\varphi(x, y)$ is a Riemann integrable function of x. Indeed,

$$\begin{aligned}\varphi(x, y) &= \Delta_{k_1\dots k_{n-1}} f(x) - \frac{1}{2}\sum_{i=1}^{n-1} \Delta_{k_1\dots k_i,k_i\dots k_{n-1}} f(y) - \\ &\quad - \Delta_{k_1\dots k_{n-1}} f(y+x) + \Delta_{k_1\dots k_{n-1}} f(y) \\ &= -\Delta_{y,k_1\dots k_{n-1}} f(x) - \frac{1}{2}\sum_{i=1}^{n-1} \Delta_{k_1\dots k_i,k_i\dots k_{n-1}} f(y) + \\ &\quad + \Delta_{k_1\dots k_{n-1}} f(y), \qquad x, y \in G.\end{aligned}$$

Since

$$\psi(x) = \int_G \varphi(x, y)\, d\mu(y), \qquad x \in G,$$

we conclude by Lemma 2.3 of [3] that ψ is Riemann integrable.

Summing up what we have shown we conclude that, for any $k_1, \dots, k_{n-1} \in G$, $\Delta_{k_1\dots k_{n-1}} g_n$ is a continuous (Riemann integrable) function.

By the induction hypothesis there exist a polynomial function $\Gamma_{n-1}\colon G \to X$ of order $n-1$ and a continuous (Riemann integrable) function $g\colon G \to X$ such that

$$g_n = \Gamma_{n-1} + g.$$

Hence

$$f = f_n + g_n = f_n + \Gamma_{n-1} + g.$$

This completes our proof.

COROLLARY 1. *Let $f\colon \boldsymbol{R} \to X$ be a* mod 1 *periodic function such that, for each $h \in \boldsymbol{R}$, $\Delta_h^n f$ is continuous (locally Riemann integrable). Then there exist a polynomial function $\Gamma\colon \boldsymbol{R} \to X$ of n-th order and a continuous (locally Riemann integrable) function $g\colon \boldsymbol{R} \to X$ such that $f = \Gamma + g$.*

Proof. Any mod 1 periodic function $f\colon \boldsymbol{R} \to X$ may be regarded as a function on the compact group $K := \boldsymbol{R}/\boldsymbol{Z}$. More precisely, if $\varkappa\colon \boldsymbol{R} \to K$ is a natural homomorphism, then $\tilde{f}\colon K \to X$ is well defined by

$$\tilde{f}(\varkappa(x)) := f(x), \qquad x \in \boldsymbol{R}.$$

It is easy to check that, for each $h \in K$, $\Delta_h^n \tilde{f}$ is continuous (Riemann integrable) if so is $\Delta_h^n f$ for each $h \in \boldsymbol{R}$. Theorems 2 and 3 yield the decomposition $\tilde{f} = \tilde{\Gamma} + \tilde{g}$ with a polynomial function $\tilde{\Gamma}\colon K \to X$ of n-th order and a continuous (Riemann integrable) function $\tilde{g}\colon K \to X$. Again, $\tilde{\Gamma}$ and $\tilde{g}$ generate functions Γ and g on $\boldsymbol{R}$ by

$$\Gamma(x) := \tilde{\Gamma}(\varkappa(x)), \qquad g(x) := \tilde{g}(\varkappa(x)), \qquad x \in \boldsymbol{R}.$$

Then $f = \Gamma + g$, which was to be shown.

3. Difference property on locally compact groups. Now, using the structural theory of locally compact groups we are going to prove the difference property of any order for the class of all continuous functions defined on an arbitrary locally compact Abelian group. In the first step we obtain the following

LEMMA 5. *For each $n \in \boldsymbol{N}$ the class of all continuous Banach-valued functions defined on $\boldsymbol{R}$ has the difference property of n-th order.*

Proof. The assertion of our lemma holds true for $n = 0$. Assume it is true for some $n-1 \in \boldsymbol{N}$ and consider an $f\colon \boldsymbol{R} \to X$ with the property that for any $h_1, \ldots, h_n \in \boldsymbol{R}$

(9) $\quad \Delta_{h_1 \ldots h_n} f$ is continuous.

Without loss of generality we can suppose that $f(0) = f(1)$. Otherwise, it is enough to prove the lemma for the function f_1:

$$f_1(x) := f(x) - [f(1) - f(0)]\, x, \qquad x \in \boldsymbol{R}.$$

Now, we define the mod 1 periodic function f^*: $\boldsymbol{R} \to X$ by

$$f^*(x) := f(x), \qquad x \in [0, 1),$$
$$f^*(x+1) = f^*(x), \qquad x \in \boldsymbol{R}.$$

We are going to show that for any $h_1, \ldots, h_{n-1} \in \boldsymbol{R}$

(10) $\quad \Delta_{h_1 \ldots h_{n-1}}(f - f^*)$ is continuous.

For, let us fix a $k \in \boldsymbol{Z}$ and choose $x_0, x \in [k, k+1)$, x tending to x_0. Then, by virtue of (9), we have

$$\begin{aligned}\Delta_{h_1 \ldots h_{n-1}}&(f-f^*)(x) - \Delta_{h_1 \ldots h_{n-1}}(f-f^*)(x_0)\\ &= \Delta_{h_1 \ldots h_{n-1}}\big(f(x) - f(x-k)\big) - \Delta_{h_1 \ldots h_{n-1}}\big(f(x_0) - f(x_0 - k)\big)\\ &= -\Delta_{h_1 \ldots h_{n-1}, -k} f(x) + \Delta_{h_1 \ldots h_{n-1}, -k} f(x_0) \to 0 \quad \text{as } x \to x_0.\end{aligned}$$

Now, let $x_0 = k+1$, $x \in (k, k+1)$, and let $x \to x_0$. Then

$$f\big(x_0 - (k+1)\big) = f(0) = f(1) = f(x_0 - k)$$

and, consequently,

$$\begin{aligned}\Delta_{h_1 \ldots h_{n-1}}&(f-f^*)(x) - \Delta_{h_1 \ldots h_{n-1}}(f-f^*)(x_0)\\ &= \Delta_{h_1 \ldots h_{n-1}}\big(f(x) - f(x-k)\big) - \Delta_{h_1 \ldots h_{n-1}}\big(f(x_0) - f(x_0 - (k+1))\big)\\ &= -\Delta_{h_1 \ldots h_{n-1}, -k} f(x) + \Delta_{h_1 \ldots h_{n-1}, -k} f(x_0) \to 0 \quad \text{as } x \to x_0.\end{aligned}$$

By the induction hypothesis we can write

$$f - f^* = \tilde{\Gamma} + \tilde{g}$$

for some continuous function $\tilde{g}$: $\boldsymbol{R} \to X$ and a polynomial function $\tilde{\Gamma}$: $\boldsymbol{R} \to X$ of order $n-1$.

From (9) and (10) it follows that for any $h_1, \ldots, h_n \in \boldsymbol{R}$ the function $\Delta_{h_1 \ldots h_n} f^*$ is continuous. Hence, applying Corollary 1, we obtain $f^* = \Gamma^* + g^*$, where Γ^*: $\boldsymbol{R} \to X$ is a polynomial function of n-th order and g^*: $\boldsymbol{R} \to X$ is continuous. Thus

$$f = (f - f^*) + f^* = (\tilde{\Gamma} + \Gamma^*) + (\tilde{g} + g^*),$$

which completes the proof by induction.

In the next step we are going to establish the difference property of an arbitrary order for the class of all continuous functions defined on the direct sum of two groups knowing that the class of all continuous functions on each of the two groups admits difference properties of any orders. For this purpose we need the following result which is due to Namioka (cf. [9], Theorem 1.2):

LEMMA 6. *Let X be a locally compact Hausdorff space, let Y be a locally* *compact and σ-compact space and suppose Z is a metric space. If a map f: $X \times Y \to Z$ is continuous in each variable separately, then there exists a dense G_δ-set $A \subset X$ such that f is jointly continuous at each point of $A \times Y$.*

Now, we can prove

LEMMA 7. *Let $(G_1, +)$ and $(G_2, +)$ be two locally compact Abelian groups and let either of them be σ-compact. Suppose that the class of all continuous Banach-valued functions on G_i has the difference property of any order for $i = 1, 2$. Then, for each $n \in N$, the class of all continuous Banach-valued functions defined on the direct sum $G_1 \oplus G_2$ has the difference property of n-th order.*

Proof. The case $n = 0$ is obvious. Suppose the assertion is true for some $n-1 \in N$.

Let f: $G_1 \oplus G_2 \to X$ be such that for any $h_1, \ldots, h_n \in G_1 \oplus G_2$ the function $\Delta_{h_1 \ldots h_n} f$ is continuous on $G_1 \oplus G_2$. If we set

$$\varphi(x) := f(x, 0), \quad x \in G_1,$$
$$\psi(y) := f(0, y), \quad y \in G_2,$$

then for each $u \in G_1$ and $v \in G_2$ we have

$$\Delta_u^n \varphi(x) = \Delta_{(u,0)}^n f(x, 0), \quad x \in G_1,$$
$$\Delta_v^n \psi(y) = \Delta_{(0,v)}^n f(0, y), \quad y \in G_2,$$

getting the continuity of $\Delta_u^n \varphi$ and $\Delta_v^n \psi$. Our assumptions guarantee the existence of polynomial functions Γ_1: $G_1 \to X$ and Γ_2: $G_2 \to X$ both of n-th order and continuous functions g_1: $G_1 \to X$ and g_2: $G_2 \to X$ such that

$$\varphi(x) = \Gamma_1(x) + g_1(x), \quad x \in G_1,$$
$$\psi(y) = \Gamma_2(y) + g_2(y), \quad y \in G_2.$$

Let us define f^*: $G_1 \oplus G_2 \to X$ by the formula

$$\begin{aligned} f^*(x, y) &:= f(x, y) - \varphi(x) - \psi(y) \\ &= f(x, y) - f(x, 0) - f(0, y), \quad (x, y) \in G_1 \oplus G_2. \end{aligned}$$

For arbitrarily fixed $h_1, \ldots, h_{n-1} \in G_1 \oplus G_2$ we consider the function ξ: $G_1 \oplus G_2 \to X$ given by

$$\xi := \Delta_{h_1 \ldots h_{n-1}} f^*.$$

From the following two equalities we obtain immediately the continuity of ξ in each variable separately:

$$\begin{aligned} \xi(x, y) &:= \Delta_{h_1 \ldots h_{n-1}} f^*(x, y) \\ &= (\Delta_{h_1 \ldots h_{n-1}} \Delta_{(0,y)} f(x, 0)) - \Delta_{h_1 \ldots h_{n-1}} f(0, y) \\ &= (\Delta_{h_1 \ldots h_{n-1}} \Delta_{(x,0)} f(0, y)) - \Delta_{h_1 \ldots h_{n-1}} f(x, 0). \end{aligned}$$

From Lemma 6 it follows in particular that there exists a point $z_0 \in G_1 \oplus G_2$ at which ξ is (jointly) continuous.

Now, it is easy to prove that ξ is continuous everywhere on $G_1 \oplus G_2$. In order to check the continuity of ξ at a point $z_1 \in G_1 \oplus G_2$ note that $\Delta_{z_1-z_0}\xi$ is a continuous function, which is readily seen from the following expansion:

$$\Delta_{z_1-z_0}\xi(x, y) = \Delta_{z_1-z_0,h_1\ldots h_{n-1}} f(x, y) - \\ - \Delta_{z_1-z_0,h_1\ldots h_{n-1}} f(x, 0) - \Delta_{z_1-z_0,h_1\ldots h_{n-1}} f(0, y).$$

Finally,

$$\xi(z_1+h) = \Delta_{z_1-z_0}\xi(z_0+h) + \xi(z_0+h) \to \Delta_{z_1-z_0}\xi(z_0) + \xi(z_0) = \xi(z_1) \quad \text{as } h \to 0.$$

Thus, for arbitrarily fixed $h_1, \ldots, h_{n-1} \in G_1 \oplus G_2$ we have obtained the continuity of $\Delta_{h_1\ldots h_{n-1}} f^*$.

By the induction hypothesis it is possible to find a polynomial function $\Gamma^*\colon G_1 \oplus G_2 \to X$ of order $n-1$ and a continuous function $g^*\colon G_1 \oplus G_2 \to X$ such that $f^* = \Gamma^* + g^*$.

Now, we can write

$$f(x, y) = f^*(x, y) + \varphi(x) + \psi(y) \\ = \Gamma^*(x, y) + g^*(x, y) + \Gamma_1(x) + g_1(x) + \Gamma_2(y) + g_2(y), \\ (x, y) \in G_1 \oplus G_2.$$

Put

$$\Gamma(x, y) := \Gamma^*(x, y) + \Gamma_1(x) + \Gamma_2(y), \quad (x, y) \in G_1 \oplus G_2,$$

$$g(x, y) := g^*(x, y) + g_1(x) + g_2(y), \quad (x, y) \in G_1 \oplus G_2.$$

Evidently, g is continuous on $G_1 \oplus G_2$, and by the identity

$$\Delta_{(u,v)}^{n+1} \Gamma(x, y) = \Delta_{(u,v)}^{n+1} \Gamma^*(x, y) + \Delta_u^{n+1} \Gamma_1(x) + \Delta_v^{n+1} \Gamma_2(y), \\ (u, v), (x, y) \in G_1 \oplus G_2,$$

Γ is a polynomial function of n-th order. The induction argument completes the proof.

The main structure theorem for compactly generated Abelian groups states (see [10], pp. 98, 99, and 110) that every Abelian group generated by a compact neighbourhood of zero is the direct sum $G_1 \oplus G_2$, where G_1 is a compact group and G_2 is an elementary group of the form $\boldsymbol{R}^a \oplus \boldsymbol{Z}^b$ ($a, b \in \boldsymbol{N}$). So, by virtue of Theorem 2, Lemma 5, and Lemma 7 applied sufficiently many times, we are able to establish difference properties of arbitrary orders for the class of all continuous Banach-valued functions on a compactly generated Abelian group. It remains to extend this result to the general case of locally compact Abelian groups. For, we need still some tools.

LEMMA 8. *If* $(G, +)$ *is an Abelian group and* H *is a subgroup of* G, *then there exists a mapping* $\omega\colon G/H \to G$ *such that*

(i) ω *is additive, i.e.,* $\omega(A+B) = \omega(A)+\omega(B)$ *for all* $A, B \in G/H$;

(ii) $\omega(A) \in A$ *for all* $A \in G/H$.

Proof. Consider the family $\mathscr{R}$ of all couples (F, ω), where F is a subgroup of G/H, $\omega\colon F \to G$ is additive, and $\omega(A) \in A$ for any $A \in F$. We order $\mathscr{R}$ assuming that (F_1, ω_1) precedes (F_2, ω_2) if and only if F_1 is a subgroup of F_2 and $\omega_2|_{F_1} = \omega_1$. Let $\mathscr{C}$ be a chain in $\mathscr{R}$. If we put

$$\tilde{F} := \bigcup\{F\colon (F, \omega) \in \mathscr{C}\}$$

and determine $\tilde{\omega}\colon \tilde{F} \to G$ by $\tilde{\omega}(A) := \omega(A)$ for $A \in F$ such that $(F, \omega) \in \mathscr{C}$, then we easily get $(\tilde{F}, \tilde{\omega}) \in \mathscr{R}$. By the Kuratowski–Zorn Lemma there exists in $\mathscr{R}$ a maximal element (F_0, ω_0) dominating $(\{H\}, 0) \in \mathscr{R}$. If it were $F_0 \neq G/H$, we would be able to find a $B \in G/H$ such that $B \notin F_0$. Let F_0' be the subgroup generated by F_0 and $\{B\}$. Choose an element b from B and define $\omega_0'\colon F_0' \to G$ by

$$\omega_0'(A+kB) := \omega_0(A)+kb, \qquad A \in F_0,\ k \in \mathbf{Z}.$$

Evidently, (F_0', ω_0') belongs to $\mathscr{R}$ and it strictly dominates the couple (F_0, ω_0), which contradicts the maximality of (F_0, ω_0). Thus $F_0 = G/H$ and the proof is complete.

LEMMA 9. *Let* $(G, +)$ *be an Abelian topological group and let* H *be an open subgroup of* G. *Suppose that the class of all continuous Banach-valued functions defined on* H *has the difference property of each order. Then for any* $n \in \mathbf{N}$ *the class of all continuous Banach-valued functions defined on* G *has the difference property of* n*-th order.*

Proof. Our lemma holds true for $n = 0$. Suppose the same for some $n-1 \in \mathbf{N}$ and consider an $f\colon G \to X$ such that for any $h_1, \ldots, h_n \in G$ the function $\Delta_{h_1 \ldots h_n} f$ is continuous on G.

Let us choose an additive mapping $\omega\colon G/H \to G$ with the property that $\omega(A) \in A$ for all $A \in G/H$. Such a choice is possible by virtue of Lemma 8. Since any x from G belongs to some coset of H, the formula

$$f^*(x) := f(x-\omega(A)) \quad \text{if } x \in A \text{ for some } A \in G/H$$

determines a function $f^*\colon G \to X$.

Observe that for any $h_1, \ldots, h_{n-1} \in G$ the function $\Delta_{h_1 \ldots h_{n-1}}(f-f^*)$ is continuous on G. Indeed, since any coset of H is open, it is enough to show that all restrictions of $\Delta_{h_1 \ldots h_{n-1}}(f-f^*)$ to cosets of H are continuous. For, choose an $A \in G/H$, take $x_0, x \in A$ and allow x to tend to x_0. Then

$$\begin{aligned} &\Delta_{h_1 \ldots h_{n-1}}(f-f^*)(x) - \Delta_{h_1 \ldots h_{n-1}}(f-f^*)(x_0) \\ &\quad = \Delta_{h_1 \ldots h_{n-1}}\big(f(x)-f(x-\omega(A))\big) - \Delta_{h_1 \ldots h_{n-1}}\big(f(x_0)-f(x_0-\omega(A))\big) \\ &\quad = -\Delta_{h_1 \ldots h_{n-1}, -\omega(A)} f(x) + \Delta_{h_1 \ldots h_{n-1}, -\omega(A)} f(x_0) \to 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as } x \to x_0,\ x \in A. \end{aligned}$$

In view of the induction hypothesis we have $f-f^* = \tilde{\Gamma}+\tilde{g}$, where $\tilde{\Gamma}$: $G \to X$ is a polynomial function of order $n-1$ and $\tilde{g}$: $G \to X$ is a continuous function. Let $f_0 := f^*|_H = f|_H$. Then, evidently, $\Delta_{h_1 \dots h_n} f_0$ is continuous on H for any fixed $h_1, \dots, h_n \in H$. By the assumptions of our lemma, we can find a polynomial function Γ_0: $H \to X$ of n-th order and a continuous function g_0: $H \to X$ such that $f_0 = \Gamma_0 + g_0$. Let us define Γ^*: $G \to X$ and g^*: $G \to X$ by the following formulae:

$$\Gamma^*(x) := \Gamma_0(x-\omega(A)) \quad \text{if } x \in A \text{ for some } A \in G/H,$$
$$g^*(x) := g_0(x-\omega(A)) \quad \text{if } x \in A \text{ for some } A \in G/H.$$

Using again the fact that all the cosets of H form an open covering of G, one can easily verify that g^* is continuous on the whole G.

Now, we shall show that Γ^* is a polynomial function of n-th order. Let $x, y \in G$, $x \in A$, $y \in B$ for some $A, B \in G/H$. Then for any $j \in \mathbf{Z}$ we have $x+jy \in A+jB$ and

$$\Gamma^*(x+jy) = \Gamma_0(x+jy-\omega(A+jB)) = \Gamma_0\big((x-\omega(A))+j(y-\omega(B))\big).$$

Hence

$$\begin{aligned}\Delta_y^{n+1}\Gamma^*(x) &= \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\Gamma^*(x+jy)\\ &= \sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}\Gamma_0\big((x-\omega(A))+j(y-\omega(B))\big)\\ &= \Delta_{y-\omega(B)}^{n+1}\Gamma_0(x-\omega(A)) = 0.\end{aligned}$$

Finally, note that $f^* = \Gamma^*+g^*$; indeed, if $x \in A$ for some $A \in G/H$, we have

$$\begin{aligned}f^*(x) &= f(x-\omega(A)) = f_0(x-\omega(A)) = \Gamma_0(x-\omega(A))+g_0(x-\omega(A))\\ &= \Gamma^*(x)+g^*(x).\end{aligned}$$

Thus, $f = (\tilde{\Gamma}+\Gamma^*)+(\tilde{g}+g^*)$ and the proof is complete by induction.

Now, we can prove the main theorem of this section.

THEOREM 4. *If $(G, +)$ is a locally compact Abelian group, then for each $n \in \mathbf{N}$ the class of all continuous Banach-valued functions defined on G has the difference property of n-th order.*

Proof. Let $U \subset G$ be an open neighbourhood of zero with the compact closure. Consider a subgroup H of G generated by U. Obviously, H is an open and compactly generated group. Lemma 9 and the considerations following the proof of Lemma 7 complete the proof of the theorem.

Remark. One cannot expect that the class of all continuous functions on any Abelian topological group which is not necessarily locally compact has difference properties of arbitrary orders. Carroll [3] has noted that it fails to hold even for the difference property of the first order.

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