COLLOQUIUM MATHEMATICUM

VOL. XXXIII 1975 FASC. 2

SOLUTION OF CAUCHY'S FUNCTIONAL EQUATION ON A RESTRICTED DOMAIN

BY

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1. Kuczma has researched in [2] the equation

$$f(x+y) = f(x)+f(y) \quad \text{for } x \in G,\ y \in H \tag{1}$$

for functions $f\colon G \to K$, where G and K are arbitrary groups (written additively), and H is a subgroup of G.

The author of [2] has given the general solution of equation (1) under the assumption that every homomorphism $g\colon H \to K$ can be extended to a homomorphism $\tilde{g}\colon G \to K$. We shall determine the general solution of equation (1) without any assumption on H. This answers a question of M. Kuczma (¹).

We start with the following obvious

LEMMA 1. *A function $f\colon G \to K$ satisfies* (1) *iff there exists a homomorphism $g\colon H \to K$ such that the following equalities are satisfied:*

$$f(x+y) = f(x)+g(y) \quad \textit{for } x \in G,\ y \in H, \tag{2}$$

$$f(y) = g(y) \quad \textit{for } y \in H. \tag{3}$$

Now we give the general solution of equation (2).

THEOREM 1. *Let $g\colon H \to K$ be a homomorphism. A function $f\colon G \to K$ is a solution of equation* (2) *iff it is of the form*

$$f(x) = h(u)+g(-u+x) \quad \textit{for } x \in u+H,\ u \in U, \tag{4}$$

where $U \subset G$ is a selector of the family $\{x+H\}_{x \in G}$ (i.e., for every $x \in G$, $U \cap (x+H)$ is a one-element set), and $h\colon U \to K$ is an arbitrary function.

(¹) Asked at the Conference on Cauchy's functional equation, Jaszowiec, May 1973.

Proof. Let $f\colon G \to K$ satisfy (2) and let U be an arbitrary selector of the family $\{x+H\}_{x\in G}$. We put

$$h(u) = f(u) \quad \text{for } u\in U. \tag{5}$$

Let $x\in u+H$, where $u\in U$. Then $-u+x\in H$, so that, by (2) and (5), we obtain

$$f(x) = f\big(u+(-u+x)\big) = f(u)+g(-u+x) = h(u)+g(-u+x).$$

Thus f can be represented in form (4).

Conversely, suppose that a function $f\colon G \to K$ is of form (4). Let $x\in G$, $y\in H$ and take $u\in U$ so that $x\in u+H$. Then also $x+y\in u+H$. Using (4), we obtain

$$\begin{aligned} f(x+y) &= h(u)+g(-u+x+y) = h(u)+g(-u+x)+g(y)\\ &= f(x)+g(y) \end{aligned}$$

which completes the proof.

COROLLARY 1. *A function $f\colon G \to K$ is a solution of equation* (1) *iff it is of form* (4) *and h satisfies additionally*

$$h(u_0) = g(u_0), \quad \textit{where } \{u_0\} = U\cap H. \tag{6}$$

Proof. By Lemma 1 and Theorem 1, it is enough to prove that, for any function $f\colon G \to K$ of form (4), where $g\colon H \to K$ is a homomorphism, (3) is equivalent to (6). Obviously, (4) yields

$$f(x) = h(u_0)+g(-u_0+x) \quad \text{for } x\in H.$$

Thus (3) is equivalent to the equality

$$g(x) = h(u_0)+g(-u_0+x) \quad \text{for } x\in H,$$

which holds iff h satisfies (3).

COROLLARY 2. *Let $g\colon H \to K$ be a homomorphism which can be extended to a homomorphism $\tilde{g}\colon G \to K$. Then a function $f\colon G \to K$ is a solution of equation* (2) *iff it can be represented in the form*

$$f(x) = \tilde{h}(x)+\tilde{g}(x) \quad \textit{for } x\in G, \tag{7}$$

where $\tilde{h}\colon G \to K$ is an arbitrary function which is constant on the left cosets of H in G.

Proof. Clearly, any function f of form (7) is a solution of equation (2). Conversely, suppose that a function $f\colon G \to K$ satisfies (2). Then, by assumption, Theorem 1 yields

$$\begin{aligned} f(x) &= h(u)+g(-u+x) = h(u)+\tilde{g}(-u+x)\\ &= h(u)+\tilde{g}(-u)+\tilde{g}(x) \quad \text{for } x\in u+H,\ u\in U. \end{aligned}$$

Putting $\tilde{h}(x) = h(u)+\tilde{g}(-u)$ for $x \in u+H$, $u \in U$, we obtain (7).

Using Lemma 1 and Corollaries 1 and 2, we can reprove the following result, obtained by Kuczma in [2], p. 314:

COROLLARY 3. *If every homomorphism g_0: $H \to K$ can be extended to a homomorphism $\tilde{g}_0$: $G \to K$, then the general solution of equation* (1) *can be written in the form*

$$f(x) = h(x)+g(x) \quad \textit{for } x \in G,$$

where g: $G \to K$ is an arbitrary homomorphism and h: $G \to K$ is an arbitrary function which is constant on the left cosets of H in G and $h(y) = 0$ for $y \in H$.

2. We are going to show further that if G is an abelian group and f: $G \to K$ satisfies equation (2) (equation (1), respectively) in a suitable subset of $G \times H$, then f satisfies equation (2) (equation (1), respectively) in $G \times H$. We adopt the following

Definition (cf. [1], p. 61, for abelian case). A non-empty family $\mathscr{I}$ of subsets of a group G is called an *invariant proper ideal* if

(i) $S_1, S_2 \in \mathscr{I}$ implies $S_1 \cup S_2 \in \mathscr{I}$;

(ii) $S_1 \in \mathscr{I}$ and $S_2 \subset S_1$ imply $S_2 \in \mathscr{I}$;

(iii) $G \notin \mathscr{I}$;

(iv) for every $x \in G$ and $S \in \mathscr{I}$, we have $x+S \in \mathscr{I}$ and $x-S \in \mathscr{I}$.

The following lemma has been suggested to the author by R. Ger:

LEMMA 2. *Let $\mathscr{I}$ be an invariant proper ideal of H and let $S \in \mathscr{I}$. Then, for any $y \in H$, there exist $y_1, y_2 \in H \setminus S$ such that $y = y_1+y_2$.*

Proof. By (iv), (i) and (iii), we have $S \cup (-S+y) \neq H$. Hence there exists a y_2 with $y_2 \in H \setminus S$ and $y_2 \notin -S+y$. Putting $y_1 = y-y_2$, we get $y_1 \in H \setminus S$ and $y = y_1+y_2$.

LEMMA 3. *Let $\mathscr{I}$ be an invariant proper ideal of H, let $S \in \mathscr{I}$, and let g: $H \to K$ be a homomorphism. If a function f: $G \to K$ satisfies the equation*

$$f(x+y) = f(x)+g(y) \quad \textit{for } x \in G \setminus H,\ y \in H \setminus S, \tag{8}$$

then

$$f(x+y) = f(x)+g(y) \quad \textit{for } x \in G \setminus H,\ y \in H.$$

Proof. Let $x \in G \setminus H$ and $y \in H$. Take y_1 and y_2 as in Lemma 2. Then $x+y_1 \in G \setminus H$. Now, using (8), we obtain

$$\begin{aligned} f(x+y) &= f(x+y_1+y_2) = f(x+y_1)+g(y_2) \\ &= f(x)+g(y_1)+g(y_2) = f(x)+g(y_1+y_2) = f(x)+g(y) \end{aligned}$$

which completes the proof.

De Bruijn has proved (cf. [1], Theorem 2) a result which can be formulated as follows:

LEMMA 4. *Let H be an abelian group, let $\mathcal{J}$ be an invariant proper ideal of H, and let $S \in \mathcal{J}$. If a function f: $H \to K$ satisfies the equation*

$$f(x+y) = f(x)+f(y) \quad \textit{for } x \in H \setminus S,\ y \in H \setminus S,$$

then

$$f(x+y) = f(x)+f(y) \quad \textit{for } x \in H,\ y \in H.$$

Now we shall prove the following

THEOREM 2. *Let H be an abelian subgroup of a group G and let $g\colon H \to K$ be a homomorphism. Let $\mathcal{J}$ be an invariant proper ideal of H, let $S \in \mathcal{J}$ and suppose there exists an $x_0 \in H \setminus S$ such that, for all x, $y \in H \setminus S$,*

$$x+y \in H\setminus S \quad \textit{or} \quad x_0+x \in H\setminus S \quad \textit{or} \quad x_0+y \in H\setminus S. \tag{9}$$

If a function f: $G \to K$ satisfies the equation

$$f(x+y) = f(x)+g(y) \quad \textit{for } x \in G\setminus S,\ y \in H\setminus S, \tag{10}$$

then

$$f(x+y) = f(x)+g(y) \quad \textit{for } x \in G,\ y \in H. \tag{11}$$

Proof. Let $x_0 \in H\setminus S$ be such that, for all x, $y \in H\setminus S$, assumptions (9) hold. We put

$$a(x) = -f(x_0)+f(x_0+x) \quad \text{for } x \in H. \tag{12}$$

From (10) we obtain

$$a(x) = g(x) \quad \text{for } x \in H\setminus S. \tag{13}$$

We prove that

$$a(x+y) = a(x)+a(y) \quad \text{for } x,\ y \in H\setminus S. \tag{14}$$

According to (9), there are three cases to be considered. If $x+y \in H\setminus S$, then (14) follows from (13) and from the assumption that g is a homomorphism. If $x_0+x \in H\setminus S$, then, by (12), (10) and (13),

$$\begin{aligned} a(x+y) &= -f(x_0)+f(x_0+x+y) = -f(x_0)+f(x_0+x)+g(y) \\ &= -f(x_0)+f(x_0)+g(x)+g(y) = a(x)+a(y). \end{aligned}$$

The case where $x_0+y \in H\setminus S$ is analogous to the preceding one.

Applying Lemma 4, from (14) we obtain

$$a(x+y) = a(x)+a(y) \quad \text{for } x,\ y \in H. \tag{15}$$

As we have proved in Lemma 2, $H\setminus S$ generates H. Since g is a homomorphism, this together with (13) and (15) implies that

$$a(x) = g(x) \quad \text{for } x \in H. \tag{16}$$

It follows from (12) and (16) that

$$f(x_0+x) = f(x_0)+g(x) \quad \text{for } x \in H.$$

Hence, for x, $y \in H$, we have

$$\begin{aligned} f(x+y) &= f\big(x_0+(-x_0+x+y)\big) = f(x_0)+g(-x_0+x+y) \\ &= f(x_0)+g(-x_0+x)+g(y) = f(x_0-x_0+x)+g(y) \\ &= f(x)+g(y). \end{aligned}$$

An application of Lemma 3 now completes the proof.

Remark. Let H be an abelian group and let $S \subset H$. If

(17) $$(-2x_0+S+S)\cap S = \emptyset,$$

then condition (9) is satisfied.

In fact, if $x_0+x \in S$ and $x_0+y \in S$, then, in view of (17),

$$x+y = -2x_0+(x_0+x)+(x_0+y) \in H\setminus S.$$

COROLLARY 4. *Let H be an abelian subgroup of a group G, let $\mathcal{I}$ be an invariant proper ideal of H, and let $S \in \mathcal{I}$. Suppose there exists an $x_0 \in H\setminus S$ such that* (9) *holds for all x, $y \in H\setminus S$. If a function $f\colon G \to K$ satisfies*

(18) $$f(x+y) = f(x)+f(y) \quad \textit{for } x \in G\setminus S,\ y \in H\setminus S,$$

then

(19) $$f(x+y) = f(x)+f(y) \quad \textit{for } x \in G,\ y \in H.$$

Proof. By applying Lemma 4, from (18) we obtain

$$f(x+y) = f(x)+f(y) \quad \text{for } x \in H,\ y \in H.$$

Let us put

(20) $$g(y) = f(y) \quad \text{for } y \in H.$$

Then the function g is a homomorphism. Putting (20) into (18), we get

$$f(x+y) = f(x)+g(y) \quad \text{for } x \in G\setminus S,\ y \in H\setminus S,$$

and, consequently, from Theorem 2

$$f(x+y) = f(x)+g(y) \quad \text{for } x \in G,\ y \in H.$$

Hence, by Lemma 1, (19) holds. This completes the proof.

REFERENCES

[1] N. G. de Bruijn, *On almost additive functions*, Colloquium Mathematicum 15 (1966), p. 59-63.

[2] M. Kuczma, *Cauchy's functional equation on a restricted domain*, ibidem 28 (1972), p. 313-315.

Reçu par la Rédaction le 3. 9. 1973;
en version modifiée le 10. 8. 1974