COLLOQUIUM MATHEMATICUM

VOL. LVIII 1990 FASC. 2

# RADIAL GROWTH AND VARIATION OF DIRICHLET FINITE HOLOMORPHIC FUNCTIONS IN THE DISK

BY

DAVID J. HALLENBECK (NEWARK, DELAWARE)
AND KRZYSZTOF SAMOTIJ (WROCŁAW)

Let $D$ be the family of functions $f$ holomorphic in $\Delta = \{z\colon |z| < 1\}$ such that

$$\iint_{\Delta} |f'(z)|^2 \, dx\, dy < +\infty \qquad (z = x+iy).$$

In [3], [4] and [6] it was proved by various methods that if $f \in D$, then

$$f'(re^{i\theta}) = o\left[\frac{1}{(1-r)^{1/2}}\right]$$

for almost all $\theta$. In this paper we give an elementary proof of this result as well as the analogous one for higher derivatives. We first prove that if $f \in D$, then

$$\int_0^r |f^{(k)}(te^{i\theta})| \, dt = o\left[\frac{1}{(1-r)^{(2k-3)/2}}\right]$$

for $k = 2, 3, \ldots$ and almost all $\theta$. This behaviour of the radial variation of $f^{(k)}$ is then used to prove that the radial growth of $f^{(k)}$ is $o[1/(1-r)^{(2k-1)/2}]$ for $k = 1, 2, \ldots$ and almost all $\theta$. We prove that these results are sharp. When $k = 1$, we obtain the result mentioned above. We also show how $f^{(k)}(re^{i\theta})$ tends to $\infty$ (as $r \to 1$) when we allow no "exceptional values" of $\theta$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

and set

$$\bar{M}(r, f) = \sum_{n=0}^{\infty} |a_n| r^n \qquad (0 < r < 1).$$

Note that

$$M(r, f) = \max_{|z|=r} |f(z)| \leqslant \bar{M}(r, f).$$

We prove that if $f \in D$, then

$$\bar{M}(r, f^{(k)}) = o\left[\frac{1}{(1-r)^k}\right] \quad \text{for } k = 1, 2, \ldots$$

This result is sharp.

LEMMA 1. *If $f \in D$, then*

$$\int_0^1 (1-t)^{2k-2} |f^{(k)}(te^{i\theta})|^2 t\,dt < +\infty \tag{1}$$

*for $k = 1, 2, \ldots$ and almost all $\theta$.*

Proof. Since $f \in D$, we have

$$\int_0^{2\pi} \left[\int_0^1 |f'(te^{i\theta})|^2 t\,dt\right] d\theta < +\infty,$$

and so, by the Tonelli theorem,

$$\int_0^1 |f'(te^{i\theta})|^2 t\,dt < +\infty \quad \text{for almost all } \theta,$$

and so (1) holds for $k = 1$. Parseval's equality gives

$$\int_0^{2\pi} |f^{(k)}(te^{i\theta})|^2 d\theta = 2\pi \sum_{n=k}^{\infty} [n(n-1)\ldots(n-(k-1))]^2 |a_n|^2 t^{2(n-k)}. \tag{2}$$

We recall that

$$\int_0^1 (1-t)^\alpha t^\beta dt = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \quad \text{for } \alpha > -1 \text{ and } \beta > -1$$

([1], p. 331). Applying this formula with $\alpha = 2(k-1)$ and $\beta = 2n-2k+1$ gives

$$\int_0^1 (1-t)^{2(k-1)} t^{2n-2k+1} dt = \frac{\Gamma(2k-1)}{(2n)\ldots(2n-(2k-2))} \tag{3}$$

for $n \geqslant k$. We infer from (2) and (3) that

$$\int_0^{2\pi}\int_0^1 (1-t)^{2k-2} |f^{(k)}(te^{i\theta})|^2 t\,dt\,d\theta = 2\pi \sum_{n=k}^{\infty} b_n |a_n|^2, \tag{4}$$

where

$$b_n = \frac{[n(n-1)\ldots(n-(k-1))]^2}{(2n)\ldots(2n-(2k-2))} \Gamma(2k-1).$$

We deduce from (4) that

$$\int_0^{2\pi}\Big(\int_0^1 (1-t)^{2k-2}|f^{(k)}(te^{i\theta})|^2\, t\,dt\Big)\,d\theta < +\infty. \tag{5}$$

It follows from (5) and the Tonelli theorem that (1) holds.

We next prove our main theorem.

THEOREM 1. *If $f \in D$, then*

$$\int_0^r |f^{(k)}(te^{i\theta})|\,dt = o\left[\frac{1}{(1-r)^{(2k-3)/2}}\right] \tag{6}$$

*for $k = 2, 3, \ldots$ and almost all $\theta$. Also, we have*

$$|f^{(k)}(re^{i\theta})| = o\left[\frac{1}{(1-r)^{(2k-1)/2}}\right] \tag{7}$$

*for $k = 1, 2, \ldots$ and almost all $\theta$.*

Proof. Let $k \neq 1$ be any positive integer and fix $\theta$ so that (1) holds. For this $\theta$, given $\varepsilon > 0$ there exists $r_0$ $(0 < r_0 < 1)$ such that

$$\int_{r_0}^r (1-t)^{2k-2}|f^{(k)}(te^{i\theta})|^2\,dt < \varepsilon^2 \qquad (r > r_0). \tag{8}$$

We note that

$$\int_0^r |f^{(k)}(te^{i\theta})|\,dt = \int_0^{r_0} |f^{(k)}(te^{i\theta})|\,dt + \int_{r_0}^r |f^{(k)}(te^{i\theta})|\,dt. \tag{9}$$

The Cauchy–Schwarz inequality implies that

$$\int_{r_0}^r |f^{(k)}(te^{i\theta})|\,dt \leqslant \Big(\int_{r_0}^r (1-t)^{2k-2}|f^{(k)}(te^{i\theta})|^2\,dt\Big)^{1/2}\left(\int_{r_0}^r \frac{1}{(1-t)^{2k-2}}\,dt\right)^{1/2}. \tag{10}$$

It follows from (10) that

$$\int_{r_0}^r |f^{(k)}(te^{i\theta})|\,dt < \varepsilon\frac{1}{\sqrt{2k-3}}\left[\frac{1}{(1-r)^{2k-3}} - \frac{1}{(1-r_0)^{2k-3}}\right]^{1/2}. \tag{11}$$

For this fixed $\varepsilon$ and $r_0$ choose $r > r_0$ sufficiently near 1 so that

$$\int_0^{r_0} |f^{(k)}(te^{i\theta})|\,dt < \varepsilon\frac{1}{(1-r)^{(2k-3)/2}}. \tag{12}$$

It follows from (9), (11) and (12) that for all $r$ sufficiently close to 1 we have

$$(1-r)^{(2k-3)/2}\int_0^r |f^{(k)}(te^{i\theta})|\,dt < 2\varepsilon. \tag{13}$$

Hence (6) holds.

We note that (7) follows from (6) and the inequality

$$|f^{(k)}(re^{i\theta})| \leqslant |f^{(k)}(0)| + \int_0^r |f^{(k+1)}(te^{i\theta})|\,dt \qquad (k = 1, 2, \ldots). \tag{14}$$

Remark. For $k = 1$, (7) is known to hold uniformly in Stolz angles. The arguments used to prove this fact are not so elementary as those given above and can be found in [3], [4] and [6]. It would be interesting to find an elementary proof that (7) holds uniformly in Stolz angles given the assumption that it holds radially. It is not difficult to prove this under the additional assumption that $f$ is univalent or that

$$\lim_{z \to e^{i\theta}} |f'(z)| = +\infty$$

uniformly in a Stolz angle. We do not give the proof here.

In the next theorem we show that (6) and (7) are sharp in a very strong sense. The construction is similar in some respects to one carried out in [5] by the first-named author and MacGregor.

THEOREM 2. *Let $\varepsilon$ be a positive function on* (0, 1) *such that*

$$\lim_{r \to 1^-} \varepsilon(r) = 0.$$

*Then there is a function $g$ in $D$ for which*

$$\varlimsup_{r \to 1^-} \left\{ \frac{(1-r)^{p-1/2} \min_{|z|=r} |g^{(p)}(z)|}{\varepsilon(r)} \right\} = +\infty \tag{15}$$

*for $p = 1, 2, \ldots$ Also,*

$$\varlimsup_{r \to 1^-} \left\{ \frac{(1-r)^{p-1/2} \int_0^r |g^{(p+1)}(te^{i\theta})|\,dt}{\varepsilon(r)} \right\} = +\infty \tag{16}$$

*for all $\theta$ and $p = 1, 2, \ldots$*

Proof. To prove (15) it suffices to show that such a function $g$ exists which has the following property: there is a sequence $\{r_k\}$ such that

$$0 \leqslant r_k < 1, \qquad \lim_{k \to \infty} r_k = 1$$

and, for every $\theta$ and every $p$,

$$(1-r_k)^{p-1/2} |g^{(p)}(r_k e^{i\theta})| \geqslant \frac{\varepsilon(r_k)}{6} \tag{17}$$

for all sufficiently large values of $k$. It is clear that (17) implies (15) by first applying (17) with $\varepsilon$ replaced by $\sqrt{\varepsilon}$. Let

$$f(z) = \sum_{n=1}^{\infty} \lambda_n^{1/2} \frac{1}{n} z^{\lambda_n},$$

where $\{\lambda_n\}$ is a strictly increasing sequence of positive integers, suitably selected in terms of $\varepsilon$ in a manner described below. Define

$$g(z) = \int_0^z f(\tau)\, d\tau.$$

It is easily verified that $g \in D$. We will prove that there is a sequence $r_k$ such that for every $\theta$ and for every $p$

$$(1-r_k)^{p+1/2}\, |f^{(p)}(r_k e^{i\theta})| \geqslant \frac{\varepsilon(r_k)}{6}. \tag{18}$$

Since $g^{(p)}(z) = f^{(p-1)}(z)$, (18) will imply (17). We construct inductively the sequence $\{\lambda_n\}$ as follows. Let $\lambda_1 = 4$, and if $\lambda_1, \lambda_2, \ldots, \lambda_{k-1}$ are already selected, then let $\lambda_k$ be so large that

$$\lambda_k \geqslant (\lambda_{k-1}+2)^2, \tag{19}$$

$$\varepsilon(1-1/\lambda_k) \leqslant 1/k. \tag{20}$$

The existence of such $\lambda_k$ is quite obvious.

Inequality (19) implies

$$\lambda_{k+1}(\lambda_{k+1}-1) \geqslant \frac{(k+1)^2}{n} \lambda_n(\lambda_n - 1) \tag{21}$$

for $n = 1, 2, \ldots, k$. To see this note that $\lambda_k \geqslant k$ and (19) imply that

$$\lambda_k \geqslant (k+1)^2 \quad \text{for } k = 1, 2, \ldots$$

Hence

$$\lambda_{k+1}(\lambda_{k+1}-1) \geqslant \lambda_k^2(\lambda_k^2-1) = (\lambda_k^2+\lambda_k)\,\lambda_k(\lambda_k-1),$$

and so

$$\lambda_{k+1}(\lambda_{k+1}-1) \geqslant [(k+1)^4+(k+1)^2]\,\lambda_k(\lambda_k-1) > \frac{(k+1)^2}{n}\lambda_n(\lambda_n-1)$$

for $n = 1, 2, \ldots, k$, and (21) holds.

Let $r_n = 1-1/\lambda_n$ for $n = 1, 2, \ldots$ and let $k$ be any positive integer. Then

$$f^{(p)}(r_k e^{i\theta}) = A+B+C, \tag{22}$$

where

$$A = \sum_{n=1}^{k-1} \lambda_n^{3/2}(\lambda_n - 1)\dots(\lambda_n - (p-1))\frac{1}{n}(r_k e^{i\theta})^{\lambda_n - p}, \tag{23}$$

$$B = \lambda_k^{3/2}(\lambda_k - 1)\dots(\lambda_k - (p-1))\frac{1}{k}(r_k e^{i\theta})^{\lambda_k - p} \tag{24}$$

and

$$C = \sum_{n=k+1}^{\infty} \lambda_n^{3/2}(\lambda_n - 1)\dots(\lambda_n - (p-1))\frac{1}{n}(r_k e^{i\theta})^{\lambda_n - p}. \tag{25}$$

We are only interested in $k$ large, and so assume $\lambda_k > p$. Also the sum in (23) takes place for values of $n$ for which $\lambda_n \geqslant p$. Inequality (21) and the fact that $\{\lambda_n\}$ is increasing imply that

$$\begin{aligned} |A| &\leqslant 2^p \left\{ \sum_{n=1}^{k-2} \frac{\lambda_n^{3/2}(\lambda_n - 1)\dots(\lambda_n - (p-1))}{n} \left(1 - \frac{1}{\lambda_k}\right)^{\lambda_n} \right. \\ &\quad \left. + \frac{\lambda_{k-1}^{3/2}(\lambda_{k-1} - 1)\dots(\lambda_{k-1} - (p-1))}{k-1} \left(1 - \frac{1}{\lambda_k}\right)^{\lambda_{k-1}} \right\} \\ &\leqslant 2^p \left\{ \sum_{n=1}^{k-2} \frac{\lambda_{k-1}^{3/2}(\lambda_{k-1} - 1)\dots(\lambda_{k-1} - (p-1))}{(k-1)^2} \right. \\ &\quad \left. + \frac{\lambda_{k-1}^{3/2}(\lambda_{k-1} - 1)\dots(\lambda_{k-1} - (p-1))}{k-1} \right\} \\ &\leqslant 2^p \left( \frac{k-2}{(k-1)^2} + \frac{1}{k-1} \right) \lambda_{k-1}^{3/2}(\lambda_{k-1} - 1)\dots(\lambda_{k-1} - (p-1)). \end{aligned}$$

Hence

$$|A| \leqslant 2^p \frac{2k-3}{(k-1)^2} \lambda_{k-1}^{3/2}(\lambda_{k-1} - 1)\dots(\lambda_{k-1} - (p-1)). \tag{26}$$

From (24) and (26) it follows that

$$\begin{aligned} \frac{|A|}{|B|} &\leqslant 2^p \frac{(2k-3)k}{(k-1)^2} \frac{\lambda_{k-1}^{3/2}(\lambda_{k-1} - 1)\dots(\lambda_{k-1} - (p-1))}{\lambda_k^{3/2}(\lambda_k - 1)\dots(\lambda_k - (p-1))} \frac{1}{(1 - 1/\lambda_k)^{\lambda_k - p}} \\ &\leqslant 2^p \frac{(2k-3)k}{(k-1)^2} \left( \frac{\lambda_{k-1}}{\lambda_k} \right)^{3/2} \frac{1}{(1 - 1/\lambda_k)^{\lambda_k - p}}. \end{aligned} \tag{27}$$

Inequality (19) implies that $\lambda_{k-1}/\lambda_k \to 0$ as $k \to \infty$. It follows from this fact and (27) that $A/B \to 0$ as $k \to +\infty$. Therefore,

$$|A| \leqslant \tfrac{1}{4}|B| \tag{28}$$

for all large $k$.

To estimate $|C|$ note that if $\lambda_k > p$, then for each $n \geqslant k$ we have

$$(29)\quad \frac{|\lambda_{n+1}^{3/2}(\lambda_{n+1}-1)\dots(\lambda_{n+1}-(p-1))(n+1)^{-1}(r_k e^{i\theta})^{\lambda_{n+1}-p}|}{|\lambda_n^{3/2}(\lambda_n-1)\dots(\lambda_n-(p-1))n^{-1}(r_k e^{i\theta})^{\lambda_n-p}|}$$
$$\leqslant \left[\frac{\lambda_{n+1}-\lambda_n+\lambda_n-(p-1)}{\lambda_n-(p-1)}\right]^{p+1/2}\left(1-\frac{1}{\lambda_k}\right)^{\lambda_{n+1}-\lambda_n}$$
$$\leqslant (\lambda_{n+1}-\lambda_n)^{p+1/2}\left(1-\frac{1}{\lambda_k}\right)^{\lambda_{n+1}-\lambda_n}$$

Now observe that the function $y$ defined by

$$y = x^{p+1/2}a^x \quad \text{for } x > 0,$$

where $p > 0$ and $0 < a < 1$, is decreasing for $x > x_0$, with

$$x_0 = -\frac{p+\frac{1}{2}}{\log a}.$$

Note also that (19) implies

$$(30)\quad \lambda_{n+1}-\lambda_n \geqslant \lambda_n^2 \geqslant \lambda_k^2.$$

Therefore, by (29) and (30) we have

$$(31)\quad |C| \leqslant \sum_{n=1}^{\infty}\left[\lambda_k^{2p+1}\left(1-\frac{1}{\lambda_k}\right)^{\lambda_k^2}\right]^n |B| = \frac{\lambda_k^{2p+1}(1-1/\lambda_k)^{\lambda_k^2}}{1-\lambda_k^{2p+1}(1-1/\lambda_k)^{\lambda_k^2}}|B|.$$

Since

$$\lim_{k\to\infty}\lambda_k^{2p+1}(1-1/\lambda_k)^{\lambda_k^2} = 0,$$

we have

$$(32)\quad |C| \leqslant \tfrac{1}{4}|B|$$

for all large $k$.

From (20), (22), (28), (32) and (24) we find that, for all large $k$,

$$(33)\quad |f^{(p)}(r_k e^{i\theta})| \geqslant \frac{1}{2}\frac{a_k}{k(1-r_k)^{p+1/2}},$$

where

$$a_k = r_k(-1+2r_k)\dots(1-(p-1)+(p-1)r_k)(1-1/\lambda_k)^{\lambda_k-p}.$$

Since $a_k \to 1/e$ as $k\to\infty$, we have $a_k \geqslant \frac{1}{3}$ for large $k$. This, (20) and (33) yield inequality (18).

It is clear that (16) follows from (15).

THEOREM 3. *If $f \in D$, then*

$$\bar{M}(r, f^{(k)}) = o\left[\frac{1}{(1-r)^k}\right] \tag{34}$$

*for* $k = 1, 2, \ldots$

Proof. Clearly,

$$\bar{M}(r, f^{(k)}) \leqslant \sum_{n=k}^{\infty} n^k |a_n|,$$

and so by the Cauchy–Schwarz inequality we have

$$\bar{M}(r, f^{(k)}) \leqslant \sum_{n=k}^{m-1} n^k |a_n| + (\sum_{n=m}^{\infty} n|a_n|^2)^{1/2} (\sum_{n=1}^{\infty} n^{2k-1} r^{2n})^{1/2}. \tag{35}$$

It is easily verified that

$$(\sum_{n=1}^{\infty} n^{2k-1} r^{2n})^{1/2} = O\left[\frac{1}{(1-r)^k}\right]$$

(see [7], pp. 38 and 224). Hence there exists $A > 0$ such that

$$(\sum_{n=1}^{\infty} n^{2k-1} r^{2n})^{-1/2} < \frac{A}{(1-r)^k} \qquad (0 < r < 1). \tag{36}$$

It follows from (35) and (36) that

$$(1-r)^k \bar{M}(r, f^{(k)}) \leqslant (1-r)^k \sum_{n=k}^{m-1} n^k |a_n| + A (\sum_{n=m}^{\infty} n|a_n|^2)^{1/2}. \tag{37}$$

Since

$$\lim_{m \to \infty} \sum_{n=m}^{\infty} n|a_n|^2 = 0,$$

(34) is easily deduced from (37). Specifically, given $\varepsilon > 0$ choose $m$ so that $\sum_{n=m}^{\infty} n|a_n|^2 < \varepsilon^2$. For this fixed $m$ choose $r$ sufficiently close to 1 so that

$$(1-r)^k \sum_{n=k}^{m-1} n^k |a_n| < \varepsilon.$$

We then have $(1-r)^k \bar{M}(r, f^{(k)}) < (1+A)\varepsilon$ for all $r$ sufficiently near 1, and hence (34) holds.

Remark. It follows from (34) that

$$\int_0^r |f^{(k)}(te^{i\theta})|\, dt = \begin{cases} o[\log(1-r)^{-1}] & \text{for } k = 1, \\ o[(1-r)^{-(k-1)}] & \text{for } k = 2, 3, \ldots \end{cases} \tag{38}$$

This contrasts with the almost everywhere radial variation of $f^{(k)}$ exhibited in (6). In [2] Cowling proved that

$$\bar{M}(r,f) = o[\sqrt{\log(1-r)^{-1}}] \quad \text{for } f \in D$$

and this result was proved to be sharp by Yamashita in [8]. The example of Yamashita can also be used to prove that (34) and (38) are sharp.

## *REFERENCES*

[1] T. Apostol, *Mathematical Analysis*, Addison-Wesley Publishing Company, 1974.
[2] V. F. Cowling, *A remark on bounded functions*, Amer. Math. Monthly 66 (1959), pp. 119–120.
[3] T. M. Flett, *On the radial order of a univalent function*, J. Math. Soc. Japan 11 (1959), pp. 1–3.
[4] F. W. Gehring, *On the radial order of subharmonic functions*, ibidem 9 (1957), pp. 77–79.
[5] D. J. Hallenbeck and T. H. MacGregor, *Radial growth and variation of bounded analytic functions*, Proc. Edinburgh Math. Soc. (2) 31 (1988), pp. 489–498.
[6] W. Seidel and J. L. Walsh, *On the derivatives of functions analytic in the unit circle and their radii of univalence and p-valence*, Trans. Amer. Math. Soc. 52 (1942), pp. 128–216.
[7] E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1939.
[8] S. Yamashita, *Cowling's theorem on a Dirichlet finite holomorphic function in the disk*, Amer. Math. Monthly 87 (1980), pp. 551–552.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF DELAWARE
NEWARK, DELAWARE 19716, U.S.A.

INSTITUTE OF MATHEMATICS
TECHNICAL UNIVERSITY OF WROCŁAW
WYBRZEŻE WYSPIAŃSKIEGO 27
PL-50-370 WROCŁAW, POLAND

*Reçu par la Rédaction le 1.9.1987;*
*en version définitive le 17.3.1988*