for $p \in M$, any smooth vector fields $X_1, \ldots, X_k$, and any section ω of the bundle $\Lambda^k T^*(M, C)$. The symbol $\langle \cdot \mid \cdot \rangle$ is here the standard duality between the vector spaces $\Lambda T_p(M, C)$ and $\Lambda T_p^*(M, C)$ for any $p \in M$, defined by $\langle v_1 \wedge \ldots \wedge v_k \mid w_1 \wedge \ldots \wedge w_l \rangle$ equal to 0 if $k \neq l$, and equal to $\det [w_j(v_i);\ i, j \leqslant k]$ if $k = l$ (cf. [7]). Thus we have the exterior differential $d\omega$ of a k-form ω defined by the formula

$$\overline{d\omega}(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \partial_{X_i} \bar{\omega}(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{k+1}) + \sum_{i<j} (-1)^{i+j} \bar{\omega}([X_i, X_j], X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_{j-1}, X_{j+1}, \ldots, X_{k+1}) \tag{2}$$

for smooth vector fields $X_1, \ldots, X_{k+1}$ on (M, C). Denote by $\mathrm{F}^k(M, C)$ the set of all k-forms being smooth sections of $\Lambda^k T^*(M, C)$. The operation d defined by (2) (cf. [5]) has the following properties:

(i) $d\alpha(p)(v) = v(\alpha)$ for v in $T_p(M, C)$ and $\alpha \in C$;

(ii) $d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$ for $\omega_1, \omega_2 \in \mathrm{F}^k(M, C)$;

(iii) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$ for $\omega \in \mathrm{F}^k(M, C)$, $\eta \in \mathrm{F}^l(M, C)$;

(iv) $d \circ d = 0$.

This yields by C-regularity of (M, C) that

(v) if $\omega \in \mathrm{F}^k(M, C)$, U is open in (M, C) and $\omega \mid U = 0$, then $d\omega \mid U = 0$.

For any smooth mapping

$$f\colon (M, C) \to (N, D) \tag{3}$$

and any $p \in M$ we have the tangent linear mapping

$$f_{*p}\colon T_p(M, C) \to T_{f(p)}(N, D)$$

defined by the formula $(f_{*p} v)(\beta) = v(\beta \circ f)$ for $\beta \in D(f(p))$, where $D(q) = \bigcup \{D_V;\ q \in V \in \tau_D\}$ (cf. [2]), and for any vector v in $T_p(M, C)$.

If (M, C) and (N, D) are of finite dimensions, then for any $\omega \in \mathrm{F}^k(N, D)$ we define the form $f^*\omega \in \mathrm{F}^k(M, C)$ in a way such that for any $p \in M$ and any vectors $v_1, \ldots, v_k$ in $T_p(M, C)$

$$\langle v_1 \wedge \ldots \wedge v_k \mid (f^*\omega)(p) \rangle = \langle f_{*p} v_1 \wedge \ldots \wedge f_{*p} v_k \mid \omega(f(p)) \rangle. \tag{4}$$

From (4) and (1) we get

$$\overline{f^*\omega}(X_1, \ldots, X_k)(p) = \langle f_{*p} X_1(p) \wedge \ldots \wedge f_{*p} X_k(p) \mid \omega(f(p)) \rangle \quad \text{for } p \in M.$$

For a given mapping (3) we have then the so-called f-pull-back $\omega \mapsto f^*\omega$. We remark that if $g\colon (N, D) \to (N', D')$, then for any $\theta \in F^k(N', D')$ we have $(g \circ f)^* \theta = f^* g^* \theta$.

LEMMA. *If (M, C) and (N, D) are differential spaces of finite dimensions, then for any smooth mapping* (3) *and any form $\omega \in F^k(N, D)$ we have*

$$df^* \omega = f^* d\omega. \tag{5}$$

Proof. Let $p \in M$. We have to prove that $(df^* \omega)(p) = (f^* d\omega)(p)$. The hypothesis that (N, D) is of finite dimension yields that there exist a neighbourhood $B \in \tau_D$ of $f(p)$ and smooth vector fields $V_1, \ldots, V_n$ on (N, D) such that $V_1(q), \ldots, V_n(q)$ is a base for $T_q(N, D)$, $q \in B$. Then (cf. [6]) there exist functions $\varepsilon_1, \ldots, \varepsilon_n \in D$ such that

$$V_i(f(p))(\varepsilon^j) = \delta_i^j, \quad i, j = 1, \ldots, n.$$

From continuity of functions $\partial_{V_i} \varepsilon^j$ it follows that at any point q of some neighbourhood of $f(p)$ we have the non-singular matrix

$$[V_i(q)(\varepsilon^j);\ i, j \leqslant n]. \tag{6}$$

We may assume that such a neighbourhood is B. Let us take the matrix $[\gamma_h^i(q);\ h, i \leqslant n]$ being inverse to (6). We have then

$$\gamma_h^i(q) \cdot V_i(q)(\varepsilon^j) = \delta_h^j.$$

Thus

$$\gamma_h^i \cdot \partial_{V_i} \varepsilon^j | B = \delta_h^j, \quad h, j = 1, \ldots, n. \tag{7}$$

Equalities (7) yield $\gamma_h^i \in D_B$. From D-regularity of the differential space (N, D) it follows that diminishing, if necessary, the set B we can find an extension of functions γ_h^i to functions belonging to D. Denote these extensions also by γ_h^i. Next, we set

$$E_h(q) = \gamma_h^i(q)\, V_i(q) \quad \text{for } q \in N,\ h = 1, \ldots, n.$$

We have then the smooth vector fields $E_1, \ldots, E_n$ which according to (7) satisfy the conditions

$$\partial_{E_h} \varepsilon^j | B = \delta_h^j, \quad h, j = 1, \ldots, n.$$

In particular, $E_1(q), \ldots, E_n(q)$ form a base for $T_q(N, D)$, $q \in B$. Set for $q \in N$

$$\bar{\omega}(q) = \sum_{i_1 < \ldots < i_k} \bar{\omega}(E_{i_1}, \ldots, E_{i_k})(q)\, d\varepsilon^{i_1}(q) \wedge \ldots \wedge d\varepsilon^{i_k}(q). \tag{8}$$

We have

$$d\varepsilon^j(q)(E_i(q)) = E_i(q)(\varepsilon^j) = (\partial_{E_i} \varepsilon^j)(q) = \delta_i^j \quad \text{for } q \in B.$$

Thus $d\varepsilon^j$ is a 1-form on (N, D) such that $\langle E_i(q) | d\varepsilon^j(q) \rangle = \delta_i^j$, $i, j = 1, \ldots, n$.

For any $q \in B$ we have a base $d\varepsilon^1(q), \ldots, d\varepsilon^n(q)$ for $(T_q(N, D))^*$. By (8), for $q \in B$ and $h_1 < \ldots < h_k$ we get

$$\begin{aligned}
&\langle E_{h_1}(q) \wedge \ldots \wedge E_{h_k}(q) | \tilde{\omega}(q) \rangle \\
&= \sum_{i_1 < \ldots < i_k} \bar{\omega}(E_{i_1}, \ldots, E_{i_k})(q) \langle E_{h_1}(q) \wedge \ldots \wedge E_{h_k}(q) | d\varepsilon^{i_1}(q) \wedge \ldots \wedge d\varepsilon^{i_k}(q) \rangle \\
&= \sum_{i_1 < \ldots < i_k} \bar{\omega}(E_{i_1}, \ldots, E_{i_k})(q) \det [\delta_{h_r}^{i_s};\ r, s \leqslant k] \\
&= \bar{\omega}(E_{h_1}, \ldots, E_{h_k})(q) = \langle E_{h_1}(q) \wedge \ldots \wedge E_{h_k}(q) | \omega(q) \rangle.
\end{aligned}$$

Hence $\tilde{\omega}(q) = \omega(q)$ for $q \in B$. Thus

$$\omega | B = \sum_{i_1 < \ldots < i_k} \bar{\omega}(E_{i_1}, \ldots, E_{i_k}) d\varepsilon^{i_1} \wedge \ldots \wedge d\varepsilon^{i_k} | B. \tag{9}$$

Let us remark that if $\omega | B = \omega_1 | B$, where $\omega, \omega_1 \in \mathrm{F}^k(N, D)$ and B is open in (N, D), then

$$f^* \omega | f^{-1}[B] = f^* \omega_1 | f^{-1}[B]. \tag{10}$$

Indeed, if ω and ω_1 are forms of degree 0, then $f^* \omega = \omega \circ f$ and $f^* \omega_1 = \omega_1 \circ f$, whereas for forms ω and ω_1 of degree $k \geqslant 1$ equality (10) follows directly from the definition of the f-pull-back by formula (4).

Let β be a form of degree 0 on (N, D). For any vector v in $T_p(M, C)$ we have

$$\begin{aligned}
(f^* d\beta)(p)(v) &= (d\beta)(f(p))(f_{*p} v) = (f_* v)(\beta) \\
&= v(\beta \circ f) = v(f^* \beta) = d(f^* \beta)(v).
\end{aligned}$$

Thus

$$f^* d\beta = df^* \beta. \tag{11}$$

Passing to the proof, by induction, of equality (5) we assume that (5) is satisfied for any form of degree lower than k, $k \geqslant 1$.

Let us take $\beta, \beta_1, \ldots, \beta_k \in D$ and set

$$\omega = \beta d\beta_1 \wedge \ldots \wedge d\beta_k. \tag{12}$$

We have then $f^* \omega = f^*(\beta d\beta_1 \wedge \ldots \wedge d\beta_{k-1}) \wedge f^* d\beta_k$; thus, by (iii), we get

$$\begin{aligned}
df^* \omega = {} & df^*(\beta d\beta_1 \wedge \ldots \wedge d\beta_{k-1}) \wedge f^* d\beta_k + \\
& + (-1)^{k-1} f^*(\beta d\beta_1 \wedge \ldots \wedge d\beta_{k-1}) \wedge d(f^* d\beta_k).
\end{aligned}$$

From (11) and (iv) it follows that $d(f^* d\beta_k) = ddf^* \beta_k = 0$. This yields

$$\begin{aligned}
df^* \omega &= df^*(\beta d\beta_1 \wedge \ldots \wedge d\beta_{k-1}) \wedge f^* d\beta_k \\
&= f^*\big(d(\beta d\beta_1 \wedge \ldots \wedge d\beta_{k-1}) \wedge d\beta_k\big) \\
&= f^* d(\beta d\beta_1 \wedge \ldots \wedge d\beta_{k-1} \wedge d\beta_k) = f^* d\omega.
\end{aligned}$$

Equality (5) is satisfied for ω of the form (12). Taking now any form $\omega \in \mathrm{F}^k(N, D)$ and $p \in M$ we state that there exist a neighbourhood B of $f(p)$ in (N, D), smooth vector fields $E_1, \ldots, E_n$ on (N, D), and $\varepsilon^1, \ldots, \varepsilon^n \in D$ such that (9) holds. Assuming (8) for $q \in N$ and $U = f^{-1}[B]$ we have, in turn,

$$\omega | B = \tilde{\omega} | B, \quad (d\omega) | B = (d\tilde{\omega}) | B, \quad (f^* d\omega) | U = (f^* d\tilde{\omega}) | U,$$

$$(f^* \omega) | U = \Big(\sum_{i_1 < \ldots < i_k} f^* (\tilde{\omega}(E_{i_1}, \ldots, E_{i_k}) d\varepsilon^{i_1} \wedge \ldots \wedge d\varepsilon^{i_k}) \Big) | U,$$

$$\begin{aligned}(df^* \omega) | U &= \Big(d \sum_{i_1 < \ldots < i_k} f^* (\tilde{\omega}(E_{i_1}, \ldots, E_{i_k}) d\varepsilon^{i_1} \wedge \ldots \wedge d\varepsilon^{i_k}) \Big) | U \\ &= \Big(\sum_{i_1 < \ldots < i_k} f^* d(\tilde{\omega}(E_{i_1}, \ldots, E_{i_k}) d\varepsilon^{i_1} \wedge \ldots \wedge d\varepsilon^{i_k}) \Big) | U \\ &= (f^* d\tilde{\omega}) | U = (f^* d\omega) | U.\end{aligned}$$

Finally, (5) holds for any $\omega \in \mathrm{F}^k(N, D)$, which completes the proof.

2. Singular simplices and chains. The standard *k-dimensional simplex* in the Cartesian space $\boldsymbol{R}^k$, i.e., the set of all points $(u^1, \ldots, u^k)$ such that $0 \leqslant u^i \leqslant 1$, $i = 1, \ldots, k$, and $u^1 + \ldots + u^k \leqslant 1$, will be denoted by Δ_k. Setting

$$u = (u^1, \ldots, u^k),$$

$$t_k^0(u) = 1 - \sum_{i=1}^{k} u^i, \quad \text{and} \quad t_k^i(u) = u^i, \; i = 1, \ldots, k,$$

we have the baricentric coordinates $t_k^0(u), \ldots, t_k^k(u)$ of the point u. In $\boldsymbol{R}^k$ we have the natural differential structure $\mathscr{E}_k$ which consists of all C^∞ real functions on $\boldsymbol{R}^k$. This structure induces the differential structure $\mathscr{E}_{k\Delta_k}$ of the simplex Δ_k.

Every smooth mapping $s\colon (\Delta_k, \mathscr{E}_{k\Delta_k}) \to (M, C)$ will be called a *singular k-simplex* on (M, C) or, shortly, a *k-simplex* on (M, C). The set of all k-simplices on (M, C) will be denoted by $\mathrm{S}_k(M, C)$. Every function $c\colon \mathrm{S}_k(M, C) \to \boldsymbol{R}$ such that the set of all $s \in \mathrm{S}_k(M, C)$ for which $c(s) \neq 0$ is finite will be called a *singular k-chain* on (M, C) or, shortly, a *k-chain* on (M, C). The set of all k-chains on (M, C) will be denoted by $\mathrm{C}_k(M, C)$.

For any $s \in \mathrm{S}_k(M, C)$ and any real number a we set $(as)(u) = a$ if $u = s$, and $(as)(u) = 0$ if $s \neq u \in \mathrm{S}_k(M, C)$. In such a way we have defined a k-chain as on (M, C). It is evident that any k-chain c may be written in the form

$$c = c(s_1) s_1 + \ldots + c(s_h) s_h, \tag{13}$$

where $\{s_1, \ldots, s_h\} = \{s;\; s \in \mathrm{S}_k(M, C) \text{ and } c(s) \neq 0\}$. It is convenient to write

$$c = \sum_s c(s) s$$

instead of (13). Taking any smooth mapping (3) and setting for $c \in \mathrm{C}_k(M, C)$

$$f_*(c) = \sum_s c(s) f \circ s$$

we get the mapping f: $\mathrm{C}_k(M, C) \to \mathrm{C}_k(N, D)$.

The set $\mathrm{C}_k(M, C)$ may be treated in a natural way as a linear space. We have then a functor $f \mapsto f_*$ from the category of differential spaces into the category of linear spaces.

Now, we take the standard immersions $\Delta_{k,i}$: $\Delta_k \to \Delta_{k+1}$ setting for any $u \in \Delta_k$

$$t^j_{k+1}(\Delta_{k,i}(u)) = \begin{cases} t^j_k(u) & \text{if } j < i, \\ 0 & \text{if } j = i, \\ t^{j-1}_k(u) & \text{if } j > i, \end{cases} \tag{14}$$

$i = 0, \ldots, k$ and $j = 0, \ldots, k+1$. From this definition it follows that

$$\Delta_{k+1,h} \circ \Delta_{k,i} = \Delta_{k+1,i+1} \circ \Delta_{k,h} \quad \text{if } h \leqslant i.$$

For any $s \in \mathrm{S}_k(M, C)$ we define the k-chain ∂s by the formula

$$\partial s = \sum_{i=0}^{k} (-1)^i s \circ \Delta_{k-1,i}, \tag{15}$$

which is called the *border* of s. For any k-chain c on (M, C) we define its border ∂c by the formula

$$\partial c = \sum_s c(s) \cdot \partial s. \tag{16}$$

Equalities (14)–(16) yield $\partial\partial c = 0$. Therefore, we may define the k-th singular homology group $H_k(M, C)$ of the differential space (M, C).

3. Integration of forms along the chains. Let us consider the Cartesian space $\boldsymbol{R}^k$ with the natural differential structure $\mathscr{E}_k$. At every its point u we have a base $\partial'^k_{1u}, \ldots, \partial'^k_{ku}$ of tangent vectors defined by the formulae $\partial'^k_{iu}\alpha = \partial_i \alpha(u)$ for any real function α being of class C^∞ in some neighbourhood of u; ∂_i is the partial derivation with respect to the i-th variable. Taking the mapping

$$\mathrm{id}_{\Delta_k}\colon (\Delta_k, \mathscr{E}_{k\,\Delta_k}) \to (\boldsymbol{R}^k, \mathscr{E}_k)$$

we state that there exists a unique system of vectors $\partial^k_{1u}, \ldots, \partial^k_{ku}$ of the space $T_u(\Delta_k, \mathscr{E}_{k\Delta_k})$ such that

$$\mathrm{id}_{\Delta_k *u}(\partial^k_{iu}) = \partial'^k_{iu}, \quad i = 1, \ldots, k.$$

This system is a base for $T_u(\Delta_k, \mathscr{E}_{k\Delta_k})$.

Let $s \in \mathrm{S}_k(M, C)$ and $\omega \in \mathrm{F}^k(M, C)$. The integral

$$\int_{\Delta_k} \langle \partial^k_{1u} \wedge \ldots \wedge \partial^k_{ku} | s^* \omega(u) \rangle du$$

will be denoted by $\int_s \omega$. For any $c \in \mathrm{C}_k(M, C)$ we set

$$\int_c \omega = \sum_s c(s) \int_s \omega. \tag{17}$$

The number $\int_c \omega$ defined by formula (17) will be called the *integral of the form ω along the chain c.*

THEOREM (Stokes' formula). *If (M, C) is a Hausdorff differential space of finite dimension, then for any form $\omega \in \mathrm{F}^k(M, C)$ and any $(k+1)$-chain c on (M, C) we have*

$$\int_c d\omega = \int_{\partial c} \omega.$$

Proof. By linearity of the mapping

$$\left(c \mapsto \int_c \omega\right): \mathrm{C}_k(M, C) \to \boldsymbol{R}$$

it suffices to prove that

$$\int_s d\omega = \sum_{h=0}^{k+1} (-1)^h \int_{s \circ \Delta_{k,h}} \omega$$

or, which is equivalent, that

$$\int_{i_{k+1}} \theta = \sum_{h=0}^{k+1} (-1)^h \int_{\Delta_{k,h}} \theta, \tag{18}$$

where $\theta = s^* \omega$ and $i_l(z) = z$ for $z \in \Delta_l$. The number on the right-hand side of formula (18) may be written as $\int_{\partial i_{k+1}} \theta$. Formula (18) takes then the form

$$\int_{i_{k+1}} d\theta = \int_{\partial i_{k+1}} \theta. \tag{19}$$

Formula (19) is nothing but Stokes' formula known in advanced calculus (cf. [6]). This completes the proof.

Stokes' formula allows us to consider the de Rham mapping

$$((h, w) \mapsto \langle h, w \rangle): H_k(M, C) \times H^k(M, C) \to \boldsymbol{R}$$

for a differential space (M, C) of finite dimension, where for any homology class h in $H_k(M, C)$ and any cohomology class w in $H^k(M, C)$ we set

$$\langle h, w \rangle = \int_c \omega, \quad c \in h \text{ and } \omega \in w.$$

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