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LOCAL COMPACTNESS AND LOCAL INVARIANCE OF FREE PRODUCTS OF TOPOLOGICAL GROUPS

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1. Introduction. If G and H are topological groups, their coproduct in the category of topological groups, denoted by $G*H$ and called the *free product*, is the algebraic free product of G and H equipped with the finest group topology inducing the given topologies on G and H. In [1] Graev showed that if G and H are Hausdorff topological groups, then $G*H$ exists and is Hausdorff. He did this by producing a group topology on the algebraic free product of G and H which induces the given topologies on G and H, and is Hausdorff. However, Graev's topology is, in general, coarser than the free product topology.

In [12] Ordman produced a simpler method of describing a Hausdorff group topology on the algebraic free product in the special case where G and H are locally invariant. (Recall that a topological group G is said to be *locally invariant* (or SIN-group [3]) if every neighbourhood of the identity element e contains a neighbourhood of e invariant under the inner automorphisms of G.) The topology Ordman put on the algebraic free product is locally invariant. This prompts the question: Is the free product of locally invariant groups necessarily locally invariant? (This would be the case if Ordman's topology was, in fact, the free product topology.) However, Ordman showed that $R*R$, where R denotes the usual topological group of reals, is not locally invariant. This led Ordman to ask if $G*H$ is ever locally invariant. In particular, he was unable to decide whether $T*T$, where T denotes the circle group, is locally invariant or not. Nevertheless, he did prove that if $\{G_i\colon i\in I\}$ is a family of topological groups, at least two of which are not discrete, then their free product $\prod^*_{i\in I} G_i$ is not both locally compact and locally invariant.

Our aim here is to give a reasonable description of the free product topology of $G*H$, where G and H are connected locally compact groups. We have some success with this in that our description allows us to deduce that the free product of a finite family of connected locally compact groups is (i) a k-space, (ii) a paracompact space, (iii) complete, (iv) never locally compact, and (v) never locally invariant. (Thus $T*T$ is not locally invariant!)

These results contrast with the fact (see [4] and [9]) that a free product of maximally almost periodic groups is maximally almost periodic. So we have the slightly curious situation that a free product of connected locally compact locally invariant groups is maximally almost periodic but not locally invariant. Our results here complement those of [10] where it was shown that a free product (a free abelian product) of an infinite number of non-totally disconnected groups is never locally compact.

2. Preliminaries.

Definition. Let G be a group and X a subset of G which generates it algebraically. Then $a \in G$ is said to be *of length n* with respect to X if n is the least integer N such that $a = x_1^{\varepsilon_1} \ldots x_N^{\varepsilon_N}$, where $\varepsilon_i = \pm 1$ and $x_i \in X$ for $i = 1, \ldots, N$. The set of all elements in G of length not greater than m will be denoted by $G_m(X)$.

Clearly, $G_1(X) = X \cup X^{-1}$ and $G_m(X)$, $m > 1$, is the product in G of m copies of $\{X \cup X^{-1} \cup \{e\}\}$, where e is the identity in G.

Our first two theorems, which were mentioned in [8], generalize Theorems 4, 5 and 6 of Graev [2]. Graev's proofs require only slight modification to yield our results and, therefore, proofs are omitted here.

THEOREM A. *Let G be a Hausdorff group with a compact subspace X which generates G algebraically. Further, let the topology of G be the finest group topology on G which induces the same topology on X. Then*

(i) *a subset V of G is closed if and only if $V \cap G_n(X)$ is compact for each n; consequently, G is a k-space;*

(ii) *G is a paracompact topological space (and hence a normal topological space);*

(iii) *G is complete in the sense of Weil (that is, G is complete in its left uniformity).*

THEOREM B. *Let G be a Hausdorff group with a compact subspace X which generates G algebraically. If the topology τ of G has the property that a subset V of G is closed if and only if $V \cap G_n(X)$ is compact for each n, then τ is the finest group topology on G which induces the given topology on X.*

Our next proposition describes the topology of a connected locally compact group in a manner suitable for our purposes.

PROPOSITION. *Let G be a connected locally compact group. Then there exists a compact subset X of G such that*

(i) *X generates G algebraically*;

(ii) *the topology of G is the finest group topology on G which induces the given topology on X*;

(iii) *if G is not compact, then there exists a compact subspace Y of X such that the subgroup generated algebraically by Y is (topologically) isomorphic to the group R of reals.*

Proof. By Section 4.13 of [7], G has a maximal compact subgroup K and subgroups $H_1, \ldots, H_r$, each isomorphic to R, such that any element $g \in G$ can be decomposed uniquely and continuously in the form $g = h_1 \ldots h_r k$, where $h_i \in H$ and $k \in K$. Each H_i has a subspace Z_i homeomorphic to the unit interval $[0, 1]$. Put $X = Z_1 \cup Z_2 \cup \ldots \cup Z_n \cup K$. Clearly, X is compact and generates G algebraically. Clearly, also condition (iii) is satisfied.

Let A be a subset of G such that $A \cap G_n(X)$ is compact for each n. To complete the proof we only have to show that A is closed. Since G is locally compact, it is a k-space [5]. Therefore, to show that A is closed, it suffices to prove that, for each compact subset B of G, $A \cap B$ is compact. But if B is any compact subset of G, then from the above description of the structure of G we see that $B \subseteq G_m(X)$ for some m. Since $A \cap G_m(X)$ is compact and $B \subseteq G_m(X)$, we infer that $A \cap B$ is compact, which completes the proof.

3. Results.

THEOREM 1. *Let $G^1, G^2, \ldots, G^m$ be Hausdorff groups which are generated algebraically by compact spaces $X_1, \ldots, X_m$, respectively. Further, let the topology of each G^i be the finest group topology inducing the same topology on X_i, and assume that $G = G^1 * G^2 * \ldots * G^m$ is the free product of the G^i. Then $X = X_1 \cup X_2 \cup \ldots \cup X_m$ is a compact set which generates G algebraically and has the property that a subset V of G is closed if and only if $V \cap G_n(X)$ is compact for each n. Further, G is* (i) *a k-space,* (ii) *a paracompact space, and* (iii) *complete.*

Proof. Let τ be the free product topology on G. Then τ is the finest group topology on G which induces the given topology τ^i on each G^i. Let τ_1 be the finest group topology on G which induces the same topology on X. Noting that X is compact and generates G algebraically, it suffices, by Theorem A, to show that $\tau = \tau_1$. Clearly, $\tau_1 \supseteq \tau$.

Note that, for each n, the topology of X completely determines the topology of $G_n(X)$. Therefore, τ and τ_1 induce the same topology on $G_n(X)$ and, indeed, also on $G^i_n(X_i)$ for each i.

Let V be a subset of G^i for some i. By Theorem A, V is closed in τ^i if and only if $V \cap G^i_n(X_i)$ is compact for all n. Since G is the algebraic free product of $\{G^1, G^2, \ldots, G^m\}$, we have $V \cap G^i_n(X_i) = V \cap G_n(X)$ for each n. So V is closed in τ^i if and only if $V \cap G_n(X)$ is compact for each n. Theorem A and the definition of τ_1 then yield that V is closed in τ^i if and only if V is closed in τ_1. Therefore, τ_1 induces the given topology τ^i on each G_i. Hence $\tau_1 \subseteq \tau$. Since $\tau \subseteq \tau_1$, we have $\tau = \tau_1$, as required. It follows from Theorem A that G is complete, paracompact, and a k-space.

COROLLARY 1. *Let $G^1, G^2, \ldots, G^m$ be locally compact groups with $G^1 \neq \{e\}$ and $G^2 \neq \{e\}$. If each G^i is either compact or connected, then $G = G^1 * G^2 * \ldots * G^m$ is a k-space, a paracompact space, and a complete topological group. Further, each G^i has a compact subspace X_i such that a subset V of G is closed if and only if $V \cap G_n(X)$, where $X = X_1 \cup X_2 \cup \ldots \cup X_m$, is compact for each n.*

Remark. We now turn to the problem of showing that $G^1 * G^1 * \ldots * G^m$ is never a connected locally compact group. Recall ([9] and [12]) that $G^1 * G^2 * \ldots * G^m$ is connected if and only if each G^i is connected. Further, since each G^i is a closed subgroup of $G^1 * G^2 * \ldots * G^m$, if $G^1 * G^2 * \ldots * G^m$ is locally compact, then each G^i is locally compact. So, without loss of generality we can assume that each G^i is a connected locally compact group.

THEOREM 2. *If $G^1, G^2, \ldots, G^m$ are connected locally compact groups with $G^1 \neq \{e\}$ and $G^2 \neq \{e\}$, then $G = G^1 * G^2 * \ldots * G^m$ is not locally compact.*

Proof. By Proposition 1, each G^i has a compact subspace X_i which generates it algebraically and is such that the topology of G^i is the finest group topology which induces the same topology on X_i. Put $X = X_1 \cup X_2 \cup \ldots \cup X_m$. Then, by Theorem 1, a subset V of G is closed if and only if $V \cap G_n(X)$ is compact for each n.

By (iii) of the Proposition, we can define compact subspaces Y_i of X_i as follows: $Y_i = X_i$ if G^i is compact; Y_i is a compact subspace of X_i such that $\mathrm{gp}\{Y_i\}$ is isomorphic to R if G^i is not compact. Let $H^i = \mathrm{gp}\{Y_i\}$ for each i. So each H^i is a connected locally compact locally invariant group.

Put $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_m$ and $H = \mathrm{gp}\{Y\}$. Then H is the free algebraic product of $\{H^1, \ldots, H^m\}$. We will show that H is the free product of $\{H^1, \ldots, H^m\}$. To do this we only need to verify that the topology of H is the finest group topology which induces the given topology on H^i for each i. In fact, we prove the stronger result that the topology of H is the finest group topology which induces the given topology on Y.

Using Corollary 1 and Theorem B it suffices to show that for each positive integer n there exists an integer l such that $G_n(X) \cap H \subseteq H_l(Y)$. Since H is the free algebraic product of $\{H^1, \ldots, H^m\}$ and G is the free algebraic product of $\{G^1, \ldots, G^m\}$, this reduces to the problem of verifying

(*) For each i and each positive integer n, there exists an integer l such that $G_n^i(X_i) \cap H^i \subseteq H_l^i(Y_i)$.

Suppose that (*) is false. Then there exists a set $A = \{a_1, a_2, \dots \dots, a_k, \dots\}$ of elements of H^i for some i such that $A \subseteq G^i_n(X_i)$ for some n, but $a_k \notin H^i_k(Y_i)$ for $k = 1, 2, \dots$ Clearly, $A \cap H^i_k(Y_i)$ is a finite set and is, therefore, compact for each k. So, by the Proposition and Theorem A, A is a closed subset of H^i. Since H^i is locally compact, it is a closed subset of G, and thus A is closed in H. Noting that $A \subseteq G^i_n(X_i)$, we see that A is compact. However, a similar argument yields that $A \setminus \{a_k\}$ is compact for each k. Thus A has the discrete topology. Consequently, A must be finite — a contradiction. Therefore (*) is true. Hence H is the free product of $\{H^1, \dots, H^m\}$.

Since each H^i is a connected locally compact group, Corollary 1 says that H is complete. Thus, if G is locally compact, then H is also locally compact. However, Ordman [12] showed that a free product of connected locally invariant groups is never locally compact. Therefore, G is not locally compact, and the proof is complete.

A slight extension of the proof of Theorem 2 yields

COROLLARY 2. *Let $G^1, G^2, \dots, G^m$ be locally compact groups with $G^1 \neq \{e\}$ and $G^2 \neq \{e\}$. If each G^i is either connected or compact and non-totally disconnected, then $G = G^1 * G^2 * \dots * G^m$ is not locally compact.*

We now turn to the problem of showing that, under reasonable conditions, $G^1 * G^2 * \dots * G^m$ is not a locally invariant group.

LEMMA. *For $i = 1, 2$, let G^i be a Hausdorff group with a compact subspace X_i which generates G_i algebraically. Further, let the topology of G^i be the finest group topology on G^i which induces the same topology on X_i, and assume that the following conditions are satisfied:*

(i) *G^1 is not discrete;*

(ii) *there exists a sequence $A_1, A_2, \dots, A_n, \dots$ of compact subsets of G^1 such that $A_n \supseteq A_{n-1}$ for $n > 1$ and*

$$\bigcup_{n=1}^{\infty} A_n = G^1 \setminus \{e\},$$

where e is the identity element.

Then $G = G^1 * G^2$ is not a locally invariant group.

Proof. Put $X = X_1 \cup X_2$. Then X generates G algebraically.

Let $x_1, x_3, x_5, \dots$ be a sequence of elements in G^2 and $x_2, x_4, x_6, \dots$ a sequence of elements in G^1 such that $x_n \neq e$ for any n. For each positive integer n, define the set Y_n by

$$Y_n = (x_1 x_2 \dots x_n)^{-1} A_n (x_1 x_2 \dots x_n).$$

Since A_n is compact, so Y_n is compact for each n. Noting that the underlying group structure of $G^1 * G^2$ is the algebraic free product of G^1 and G^2, we see that the length of each element in Y_n, with respect to X, is exactly $2n+1$. Thus, if we define Y by

$$Y = \bigcup_{n=1}^{\infty} Y_n,$$

we have

$$Y \cap G_n(X) = Y_1 \cup Y_2 \cup \ldots \cup Y_k \quad \text{for some } k.$$

Since each Y_i is compact, we see that $Y \cap G_n(X)$ is compact for each n. By Theorem 1 this implies that Y is a closed subset of G.

Noting that $e \notin Y_n$ for any n, we see that $G \setminus Y$ is an open neighbourhood of e.

Suppose that G is a locally invariant group. Then there exists a neighbourhood I of e such that $I \subseteq G \setminus Y$ and I is invariant under all the inner automorphisms of G. Now $I \cap G^1$ is a neighbourhood of e in G^1. Since G^1 is not discrete, there exists an element $g \in I \cap G^1$ such that $g \neq e$.

Now, by our assumption (ii), there exists an n such that $g \in A_n$. Thus

$$(x_1 x_2 \ldots x_n)^{-1} g x_1 x_2 \ldots x_n \in Y_n \subset Y.$$

Therefore

$$(x_1 x_2 \ldots x_n)^{-1} g x_1 x_2 \ldots x_n \notin I,$$

which contradicts the fact that I is invariant under all the inner automorphisms of G. Hence $G^1 * G^2$ is not a locally invariant group.

THEOREM 3. *Let $G^1, \ldots, G^m$ be locally compact groups with $G^1 \neq \{e\}$ and $G^2 \neq \{e\}$. If G^1 is either connected or compact and non-totally disconnected, and $G^2, \ldots, G^m$ are each either connected or compact, then $G^1 * G^2 * \ldots \ldots * G^m$ is not a locally invariant group.*

Proof. Suppose that $G^1 * G^2 * \ldots * G^m$ is locally invariant. Then $G^1 * G^2$, being a subgroup of $G^1 * G^2 * \ldots * G^m$, is also locally invariant.

By Section 4.6 of [7], G^1 has a quotient group H which is a non-discrete Lie group and is either compact or connected. Noting that $H * G^2$ is a quotient group of $G^1 * G^2$ [12], we infer that $H * G^2$ is locally invariant.

As usual, let X_1 and X_2 be compact subsets of H and G^2, respectively, such that they have the properties described in the Proposition. Put $X = X_1 \cup X_2$ and $G = H * G^2$. Note that $G_n(X)$ is compact for each n.

As H is a Lie group, it is metrizable. Let d be a compatible metric. Write

$$A_n = G_n(X) \cap \left\{x \colon x \in G \text{ and } d(x, e) \geqslant \frac{1}{n}\right\}$$

for each positive integer n. Now A_n, being a closed subset of $G_n(X)$, is compact, $A_n \supseteq A_{n-1}$ for $n > 1$, and

$$\bigcup_{n=1}^{\infty} A_n = H \setminus \{e\}.$$

Thus, by the Lemma, $H*G^2$ is not a locally invariant group — a contradiction.

Additional remark. Since this paper was first written, other related work has been done. In particular, we mention [6], [11], [13], and [14].

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