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PARTIAL BOOLEAN ALGEBRAS IN A BROADER SENSE AND BOOLEAN EMBEDDINGS

BY

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The present paper provides some theorems on embeddability of partial Boolean algebras in a broader sense into Boolean algebras. The subject of the article is entirely inspired by physics. The physical background of the problem under consideration and the role of partial Boolean algebras in foundations of physics are sketched in [2] and [3]. A partial Boolean algebra which can be embedded into a Boolean algebra is usually called *classical* [3]. Interdependencies between the so-called abstract classical physical systems and classical partial Boolean algebras are given in [3] (cf. also [4]).

The article has the following structure. We begin with the definition of a partial Boolean algebra in a broader sense. Some useful comments explain the origin of the notion. Further, we give the definitions of a homomorphism, of a weak embedding and of an embedding of a partial algebra into a partial algebra. Next, we reach for the notion (or rather construction) of direct limit of an ordered system of Boolean algebras. The notion is due to Dwinger [1]. The usefulness of the notion in the domain of partial Boolean algebras follows directly from the fact that any family of Boolean subalgebras of a partial Boolean algebra forms an ordered system of Boolean algebras. Using the notion of direct limit of Boolean algebras we state first the auxiliary theorem (Theorem 2) and next we prove the main theorems of the paper (Theorems 3-5).

Definition 1. By a *partial Boolean algebra in a broader sense* we shall mean the system

$$(1)\qquad \mathscr{B} = \langle B;\ \{C_n\}_{n\geqslant 2},\ \vee,\ \neg,\ \mathbf{1}\rangle$$

with the following properties: B is a set, $\mathbf{1} \in B$, for each natural integer $n \geqslant 2$ C_n is an n-ary relation in B ($C_n \subseteq B^n$) called the n-ary *commeasurability relation*, $\vee$ is a binary function mapping C_2 into B called the *partial Boolean sum* in $\mathscr{B}$, $\neg$ is a unary function mapping B into B. The function

$\neg$ is called the *Boolean complementation* in $\mathscr{B}$. Moreover, we assume that

(1°) $C_2(a, \mathbf{1})$ for all $a \in B$;

(2°) if $C_n(a_1, \ldots, a_n)$, then $C_n(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ for every permutation σ of the set $\{1, \ldots, n\}$ ($C_n(b_1, \ldots, b_n)$ means that $\langle b_1, \ldots, b_n\rangle \in C_n$);

(3°) if $C_{n+1}(a_1, \ldots, a_n, a_{n+1})$, then $C_n(a_1, \ldots, a_n)$;

(4°) if $C_n(a_1, \ldots, a_n)$, then $C_{n+1}(a_1 \vee a_2, a_1, a_2, \ldots, a_n)$;

(5°) if $C_n(a_1, \ldots, a_n)$, then $C_n(\neg a_1, a_2, \ldots, a_n)$;

(6°) if $C_3(a_1, a_2, a_3)$, then the Boolean polynomials in a_1, a_2, a_3 form a Boolean algebra.

(Condition (6°) is equivalent to a longer but more elementary one: if $C_3(a_1, a_2, a_3)$, then $a_i \vee a_j = a_j \vee a_i$, $a_i \vee (a_j \vee a_k) = (a_i \vee a_j) \vee a_k$, etc. $(1 \leqslant i, j \leqslant 3)$.)

The notion of partial Boolean algebra in a broader sense is connected with the notion of so-called compatible family of Boolean algebras. The remarks given in the sequel explain these interdependencies.

An indexed family of Boolean algebras

$$\{\mathscr{B}_\lambda\colon \lambda \in \Lambda\}, \tag{2}$$

where $\mathscr{B}_\lambda = \langle B_\lambda; \vee_\lambda, \neg_\lambda, \mathbf{1}_\lambda\rangle$, is *compatible* provided that

(i) $\mathbf{1}_\mu = \mathbf{1}_\nu$ for all $\mu, \nu \in \Lambda$;

(ii) if $a, b \in B_\mu \cap B_\nu$, then $a \vee_\mu b = a \vee_\nu b$;

(iii) if $a \in B_\mu \cap B_\nu$, then $\neg_\mu a = \neg_\nu a$.

Let (2) be a compatible family. Then

$B \stackrel{\text{df}}{=} \bigcup_{\lambda \in \Lambda} B_\lambda$;

$\mathbf{1} \stackrel{\text{df}}{=} \mathbf{1}_\lambda$, λ — arbitrary;

$\neg a \stackrel{\text{df}}{=} \neg_\lambda a$, where λ is an arbitrary element of Λ such that $a \in B_\lambda$;

$a \vee b \stackrel{\text{df}}{=} a \vee_\lambda b$, where λ is an arbitrary element of Λ such that $a, b \in B_\lambda$;

for each natural $n \geqslant 2$, C_n is an n-ary relation in B: $C_n(a_1, \ldots, a_n)$ iff there exists a $\lambda \in \Lambda$ such that $a_1, \ldots, a_n \in B_\lambda$.

Notice that $a \vee b$ exists exactly if $C_2(a, b)$.

It is immediately seen that the system $\langle B; \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ we have defined fulfills the axioms of partial Boolean algebras in a broader sense (conditions (1°)-(6°)). This system will be called the *partial Boolean algebra in a broader sense determined by a compatible family* (2) of Boolean algebras.

It turns out that any partial Boolean algebra in a broader sense is determined by a certain compatible family of Boolean algebras. To show this we need some definitions.

Definition 2 ([3]). Assume that $\mathscr{B} = \langle B; \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ is a partial Boolean algebra in a broader sense. We say that the system $\mathscr{A} = \langle A; \vee, \neg, \mathbf{1}\rangle$ is a *Boolean algebra* in $\mathscr{B}$ whenever

(i) $\mathbf{1} \in A \subseteq B$;
(ii) $A \times A \subseteq C_2$;
(iii) if $a_1, a_2 \in A$, then $a_1 \vee a_2 \in A$;
(iv) if $a \in A$, then $\neg a \in A$;
(v) if $a_1, a_2, a_3 \in A$, then

$$a_1 \vee (a_2 \vee a_3) = (a_1 \vee a_2) \vee a_3$$

and

$$a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3).$$

Obviously, every Boolean algebra in $\mathscr{B}$ is a Boolean algebra in the usual sense.

Definition 3. Let $\mathscr{B} = \langle B; \{C_n\}_{n \geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ be a partial Boolean algebra in a broader sense. We say that a Boolean algebra $\mathscr{A} = \langle A; \vee, \neg, \mathbf{1}\rangle$ in $\mathscr{B}$ *has property* C whenever for every $n \geqslant 2$

(C) if $a_1, \ldots, a_n \in A$, then $C_n(a_1, \ldots, a_n)$.

In [3], Corollary (1.1), the following theorem is proved:

THEOREM 1. *Let $\mathscr{B} = \langle B; \{C_n\}_{n \geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ be a partial Boolean algebra in a broader sense. Let A_0 be a subset of B such that $C_n(a_1, \ldots, a_n)$ for any $a_1, \ldots, a_n \in A_0$ and every $n \geqslant 2$. Then there exists a Boolean algebra $\mathscr{A} = \langle A; \vee, \neg, \mathbf{1}\rangle$ in $\mathscr{B}$ with property* C *such that $A_0 \subseteq A$.*

Now, let $\mathscr{B} = \langle B; \{C_n\}_{n \geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ be any partial Boolean algebra and let $\{\mathscr{B}_\lambda : \lambda \in \Lambda\}$ be the family of all Boolean algebras in $\mathscr{B}$ having property C. Notice that for each $a \in B$ we can find a $\lambda \in \Lambda$ such that $a \in B_\lambda$. Hence

$$B = \bigcup_{\lambda \in \Lambda} B_\lambda.$$

Define a relation C_n^* as follows:

$C_n^*(a_1, \ldots, a_n)$ iff there exists a $\lambda \in \Lambda$ such that $a_1, \ldots, a_n \in B_\lambda$.

Thus $\mathscr{B}^* = \langle B; \{C_n^*\}_{n \geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ is the partial Boolean algebra determined by the compatible family $\{\mathscr{B}_\lambda : \lambda \in \Lambda\}$.

We shall show that the partial algebras $\mathscr{B}$ and $\mathscr{B}^*$ are identical. It suffices to prove that $C_n = C_n^*$ for any $n \geqslant 2$. Let $C_n(a_1, \ldots, a_n)$ in $\mathscr{B}$. Then by Theorem 1 there is a $\lambda \in \Lambda$ such that $\{a_1, \ldots, a_n\} \subseteq B_\lambda$. Hence $C_n^*(a_1, \ldots, a_n)$. Now, let $C_n^*(a_1, \ldots, a_n)$. Then there exists a λ such that $a_1, \ldots, a_n \in B_\lambda$. But $\mathscr{B}_\lambda$ has property C. Hence $C_n(a_1, \ldots, a_n)$.

The notion of partial Boolean algebra in a broader sense is more general than the notion of partial Boolean algebra in the sense of Kochen and Specker [5]. Recall that a *partial Boolean algebra in the sense of Kochen and Specker*, denoted by

(3) $$\mathscr{B} = \langle B; \circlearrowleft, \vee, \neg, \mathbf{1}\rangle,$$

is given by a set B, a binary relation $\circlearrowleft$ on A, a binary partial function $\vee$ from $\circlearrowleft$ into B (the partial Boolean sum in $\mathscr{B}$), a unary function $\neg$

from B into B (the Boolean complementation in $\mathscr{B}$) and an element $\mathbf{1}$ of B. Moreover:

(1*) $\between$, called the *relation of commeasurability* in $\mathscr{B}$, is symmetric and reflexive;

(2*) for all $b \in B$, $b \between \mathbf{1}$ (the constant $\mathbf{1}$ is commeasurable with all elements of B);

(3*) the partial function $\vee$ is defined exactly for those pairs $\langle a, b\rangle$ of $B \times B$ for which $a \between b$;

(4*) if any two of a, b, c are commeasurable, then $(a \vee b) \between c$ and $\neg a \between b$;

(5*) if any two of a, b, c are commeasurable, then the Boolean polynomials in a, b, c form a Boolean algebra.

For a partial algebra given by (3) and for each $n \geqslant 2$ define C_n as follows:

$$C_n(a_1, \ldots, a_n) \quad \text{iff} \quad a_i \between a_j \text{ for all } i, j \ (1 \leqslant i, j \leqslant n).$$

Then the system $\langle B;\ \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ is a partial Boolean algebra in a broader sense. Notice that $C_2 = \between$. Conversely, suppose we are given a partial Boolean algebra (1) in a broader sense such that for any $n \geqslant 2$ and all $a_1, \ldots, a_n \in B$:

$$(*_n) \qquad C_n(a_1, \ldots, a_n) \quad \text{iff} \quad C_2(a_i, a_j) \text{ for all } i, j \ (1 \leqslant i, j \leqslant n).$$

Then, it is easy to check that $\langle B; C_2, \vee, \neg, \mathbf{1}\rangle$ is a partial Boolean algebra in the sense of Kochen and Specker.

It should be stressed that every Boolean algebra

$$(4) \qquad \mathscr{B} = \langle B; \vee, \neg, \mathbf{1}\rangle$$

can be treated as a partial Boolean algebra in a broader sense. For a given Boolean algebra (4), we set

$$C_n \overset{\mathrm{df}}{=} B^n = \underbrace{B \times \ldots \times B}_{n \text{ times}}, \quad n \geqslant 2.$$

Then the system

$$(5) \qquad \langle B;\ \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$$

is a partial Boolean algebra in a broader sense. Obviously, the function $\vee$ is defined for any pair $\langle a, b\rangle$, $a, b \in B$. This identification of a Boolean algebra (4) with a partial algebra (5) is assumed in the following definition:

Definition 4. Let $\mathscr{B} = \langle B;\ \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ and $\mathscr{B}' = \langle B';\ \{C'_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ be partial Boolean algebras in a broader sense. Then:

A. A mapping $h: B \to B'$ is said to be a *homomorphism* whenever

(a$_1$) if $C_n(a_1, \ldots, a_n)$, then $C'_n(ha_1, \ldots, ha_n)$ $(n \geqslant 2)$;

(a$_2$) if $C_2(a, b)$, then $h(a \vee b) = ha \vee hb$;

(a$_3$) $h(\neg a) = \neg(ha)$;

(a$_4$) $h\mathbf{1} = \mathbf{1}$.

B. A homomorphism $h\colon B \to B'$ is called a *weak embedding* if $C_2(a, b)$ and $a \neq b$ implies $ha \neq hb$ in $\mathscr{B}'$.

C. A one-to-one homomorphism is called an *embedding*.

D. An embedding $h\colon B \to B'$ of type "onto" such that $C_n(a_1, \ldots, a_n)$ iff $C'_n(ha_1, \ldots, ha_n)$ is called an *isomorphism*.

A partially ordered system of Boolean algebras (cf. [1], p. 320) is a set $\{\mathscr{B}_\lambda, f_{\lambda\mu} : \lambda, \mu \in \Lambda\}$, where Λ is a partially ordered set, $\mathscr{B}_\lambda$ is a Boolean algebra for every $\lambda \in \Lambda$, and $f_{\lambda\mu}\colon \mathscr{B}_\lambda \to \mathscr{B}_\mu$ for $\lambda \leqslant \mu$ is a homomorphism such that $f_{\lambda\lambda}$ is the identity map for every $\lambda \in \Lambda$ and $f_{\mu\nu} \circ f_{\lambda\mu} = f_{\lambda\nu}$ whenever $\lambda \leqslant \mu \leqslant \nu$.

The direct limit of a partially ordered system $\{\mathscr{B}_\lambda, f_{\lambda\mu} : \lambda, \mu \in \Lambda\}$ of Boolean algebras is a pair $(\mathscr{A}, \{i_\lambda : \lambda \in \Lambda\})$, where $\mathscr{A}$ is a Boolean algebra, and $i_\lambda : \mathscr{B}_\lambda \to \mathscr{A}$ for every $\lambda \in \Lambda$ is a homomorphism such that

(i) $i_\mu \circ f_{\lambda\mu} = i_\lambda$ whenever $\lambda \leqslant \mu$;

(ii) for every pair $(\mathscr{C}, \{h_\lambda : \lambda \in \Lambda\})$, where $\mathscr{C}$ is a Boolean algebra and $h_\lambda : \mathscr{B}_\lambda \to \mathscr{C}$ is a homomorphism such that $h_\mu \circ f_{\lambda\mu} = h_\lambda$ whenever $\lambda \leqslant \mu$, there exists a unique homomorphism $h : \mathscr{A} \to \mathscr{C}$ such that $h \circ i_\lambda = h_\lambda$ for every $\lambda \in \Lambda$.

It should be noted that direct limits always exist ([1], p. 320-322). ($\mathscr{A}$ may be a trivial algebra!)

Now assume $\mathscr{B} = \langle B; \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ to be a partial Boolean algebra in a broader sense. Let $\{\mathscr{B}_\lambda\}_{\lambda\in\Lambda}$ be the indexed family of all Boolean algebras in $\mathscr{B}$ with property C. Let, by definition, $\lambda \leqslant \mu$ iff $B_\lambda \subseteq B_\mu$, and whenever $\lambda \leqslant \mu$ let $f_{\lambda\mu}$ be the identity map from B_λ into B_μ. Then

(6) $$\{\mathscr{B}_\lambda, f_{\lambda\mu} : \lambda, \mu \in \Lambda\}$$

is an ordered system of Boolean algebras. Notice that $\langle \Lambda, \leqslant\rangle$ is a complete meet semilattice. Now, let

(7) $$(\mathscr{A}, \{i_\lambda : \lambda \in \Lambda\})$$

be the direct limit of the system (6).

Recall that $\mathscr{A}$ is constructed as follows. Let $(\mathscr{C}, \{h_\lambda : \lambda \in \Lambda\})$ be the Boolean product of the algebras $\{\mathscr{B}_\lambda\}_{\lambda\in\Lambda}$. Let $\mathscr{C}_\lambda = h_\lambda(\mathscr{B}_\lambda)$. Obviously, $\mathscr{C}_\lambda$ is a subalgebra of $\mathscr{C}$ $(\lambda \in \Lambda)$. Hence the family $\{\mathscr{C}_\lambda\}_{\lambda\in\Lambda}$ is compatible, and thus forms a partial Boolean algebra in a broader sense, to be denoted by

$$\mathscr{C}_0 = \langle C_0; \{C_n^0\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}_{\mathscr{C}}\rangle.$$

Let Δ be the ideal in $\mathscr{C}$ generated by the elements

(8) $$h_\lambda(a) \mathbin{\Delta} h_\mu \circ f_{\lambda\mu}(a),$$

where $a \in B_\lambda$, $\lambda \leqslant \mu$, and Δ is the symmetrical difference. Δ need not be a proper ideal. Let $\varphi : \mathscr{C} \to \mathscr{C}/_\Delta$ be the natural homomorphism and let $\varphi_\lambda = \varphi\restriction_{\mathscr{C}_\lambda}$ for every $\lambda \in \Lambda$. Let $i_\lambda = \varphi_\lambda \circ h_\lambda$. Then $(\mathscr{C}/_\Delta, \{i_\lambda : \lambda \in \Lambda\})$ is the direct limit of the system (6).

Define the relation $\sim$ in C_0 as follows:

$$x \sim y \quad \text{iff} \quad x = h_{\lambda_1}(a) \text{ and } y = h_{\lambda_2}(a)$$

for certain $\lambda_1, \lambda_2 \in \Lambda$ and $a \in B$.

By independency of the algebras $\mathscr{C}_\lambda$, $\lambda \in \Lambda$, if $x \neq \mathbf{0}, \mathbf{1}$, then there exist exactly one λ and exactly one a such that $x = h_\lambda(a)$. Moreover, $\sim$ is an equivalence relation in

$$C_0 = \bigcup_{\lambda \in \Lambda} h_\lambda(B_\lambda).$$

Consequently, the ideal Δ is generated by elements of the form $x\Delta y$, where $x, y \in C_0$ and $x \sim y$. Indeed, Δ is generated by (8). But for (8) we have

$$h_\lambda(a) \sim h_\mu \circ f_{\lambda\mu}(a).$$

Conversely, if $x \sim y$ and $x = h_{\lambda_1}(a)$, $y = h_{\lambda_2}(a)$, then there is a $\lambda_0 \leqslant \lambda_i$ $(i = 1, 2)$ such that $a \in B_{\lambda_0}$. Hence

$$h_{\lambda_0}(a) \,\Delta\, h_{\lambda_i} \circ f_{\lambda_0\lambda_i}(a), \qquad i = 1, 2,$$

and, consequently,

$$h_{\lambda_1} \circ f_{\lambda_0\lambda_1}(a) \,\Delta\, h_{\lambda_2} \circ f_{\lambda_0\lambda_2}(a) \in \Delta, \qquad \text{i.e.,} \qquad x\Delta y \in \Delta.$$

Let $|x_1|, \ldots, |x_n| \in C_0/_\sim$. We define the n-ary relation C'_n in $C_0/_\sim$ for all $n \geqslant 2$ as follows:

$C'_n(|x_1|, \ldots, |x_n|)$ iff there exist $a_1, \ldots, a_n \in B$ such that $x_k = h_{\lambda_k}(a_k)$ for $k = 1, \ldots, n$, and $C_n(a_1, \ldots, a_n)$ in $\mathscr{B}$.

If $C'_2(|x|, |y|)$ and $x = h_\mu(a)$, $y = h_\nu(b)$, then there is a $\lambda_0 \in \Lambda$ such that $a, b \in B_{\lambda_0}$. Let $x_0 = h_{\lambda_0}(a)$ and $y_0 = h_{\lambda_0}(b)$. Obviously, $x \sim x_0$ and $y \sim y_0$. Then

$$|x| \vee |y| \stackrel{\text{df}}{=} |x_0 \vee y_0|.$$

If $|x| \in C_0/_\sim$, then

$$\neg |x| \stackrel{\text{df}}{=} |\neg x|.$$

The relations C'_n and the functions $\vee$ and $\neg$ are well defined.

THEOREM 2 [1]. *Let $\mathscr{B} = \langle B; \{C_n\}_{n \geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ be a partial Boolean algebra in a broader sense. Let $\mathscr{C}_0$ be defined as above. Then:*

1. *The system $\mathscr{C}_0/_\sim = \langle C_0/_\sim; \{C'_n\}_{n \geqslant 2}, \vee, \neg, \mathbf{1}_{\mathscr{C}}\rangle$ is a partial Boolean algebra in a broader sense.*

2. *The function $\varphi_0 : C_0 \to C_0/_\sim$ defined by the formula $\varphi_0(x) = |x|$ maps homomorphically $\mathscr{C}_0$ onto $\mathscr{C}_0/_\sim$.*

3. *$\mathscr{B}$ and $\mathscr{C}_0/_\sim$ are isomorphic.*

We omit the easy proof.

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[1] A similar theorem for partial Boolean algebras in the sense of Kochen and Specker has been given in [2].

We admit the following notation. If $\mathscr{B}$ is a partial Boolean algebra in a broader sense and $a \in B$, then

$$(-1)a \overset{\mathrm{df}}{=} \neg a \quad \text{and} \quad (+1)a \overset{\mathrm{df}}{=} a.$$

The principal theorems of this paper, formulated in the sequel, give sufficient and necessary conditions for a partial Boolean algebra in a broader sense to be homomorphically mapped into a nontrivial Boolean algebra, weakly embeddable into a Boolean algebra and, finally, embeddable into a Boolean algebra, respectively. We have decided to present in detail only the proof of the first theorem. The proofs of the remaining theorems are similar.

THEOREM 3. *Let $\mathscr{B} = \langle B; \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ be a partial Boolean algebra in a broader sense. The following conditions are equivalent:*

(i) *There exists a homomorphism of $\mathscr{B}$ into a nontrivial Boolean algebra.*

(ii) *For every finite sequence $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ of elements of B there exists a sequence $\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(n)$, where $\varepsilon(k) = -1$ or $\varepsilon(k) = +1$ $(k = 1, 2, \ldots, n)$, such that for every subsequence $(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ of $\boldsymbol{a}$ with the property $C_m(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ the following inequality holds:*

$$\bigwedge_{i=1}^{m} \varepsilon(k_i) a_{k_i} \neq \mathbf{0}. \tag{9}$$

Proof. (i) $\Rightarrow$ (ii). Let h map homomorphically $\mathscr{B}$ into a Boolean algebra $\mathscr{A} = \langle A, \vee, \neg, \mathbf{1}\rangle$. For any finite sequence $(b_1, b_2, \ldots, b_n)$ of elements of A we can find a sequence $(\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(n))$, where $\varepsilon(k) = -1$ or $\varepsilon(k) = +1$ $(k = 1, 2, \ldots, n)$, such that

$$\bigwedge_{i=1}^{n} \varepsilon(i) b_i \neq \mathbf{0}.$$

Consequently, for any subsequence $(b_{k_1}, b_{k_2}, \ldots, b_{k_m})$ we have

$$\bigwedge_{i=1}^{m} \varepsilon(k_i) b_{k_i} \neq \mathbf{0}.$$

Now, let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ be an arbitrary sequence of elements of B. Hence there is a sequence $\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(n)$ such that

$$\bigwedge_{i=1}^{n} \varepsilon(i) h(a_i) \neq \mathbf{0} \quad \text{in } \mathscr{A}.$$

Take any subsequence $(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ of the sequence $\boldsymbol{a}$ such that $C_m(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ in $\mathscr{B}$. In particular,

$$\bigwedge_{i=1}^{m} \varepsilon(k_i) h(a_{k_i}) \neq \mathbf{0}.$$

Since h is a homomorphism and $C_m(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$, we infer that inequality (9) holds in $\mathscr{B}$.

(ii) $\Rightarrow$ (i). Let, according to the previous notation, $(\mathscr{C}/_{\Delta}, \{i_\lambda : \lambda \in \Lambda\})$ be the direct limit of the partially ordered system $\{\mathscr{B}_\lambda, f_{\lambda\mu} : \lambda, \mu \in \Lambda\}$ of all Boolean algebras in $\mathscr{B}$ with property C. We shall show that under assumption (ii) the ideal Δ is proper.

Let $u_k = x_k \Delta y_k$ $(k = 1, 2, \ldots, n)$, where $x_k \sim y_k$ $(x_k, y_k \in C_0$ and $k = 1, 2, \ldots, n)$. Assume that $x_k = h_{\mu_k}(a_k)$ and $y_k = h_{\nu_k}(a_k)$. Form the sequence $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$. Then there exist $\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(n)$ such that

$$\varepsilon(k_1)a_{k_1} \wedge \ldots \wedge \varepsilon(k_m)a_{k_m} \neq \mathbf{0}$$

for every subsequence $(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ of the sequence $\boldsymbol{a}$ with the property $C_m(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$.

Let Λ_0 be the set of elements of Λ appearing in the sequences $(\mu_1, \mu_2, \ldots, \mu_n)$ and $(\nu_1, \nu_2, \ldots, \nu_n)$. Then for every $\lambda \in \Lambda_0$

$$\mathbf{0} \neq b_\lambda \overset{\mathrm{df}}{=} \varepsilon(k_1)a_{k_1} \wedge \ldots \wedge \varepsilon(k_m)a_{k_m},$$

where $(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ is the subsequence of $\boldsymbol{a}$ consisting of elements belonging to the algebra $\mathscr{B}_\lambda$. Hence

$$\mathbf{0}_{\mathscr{C}} \neq h_\lambda(b_\lambda) = \varepsilon(k_1)h_\lambda(a_{k_1}) \wedge \ldots \wedge \varepsilon(k_m)h_\lambda(a_{k_m})$$

in the algebra $\mathscr{C}_\lambda = h_\lambda(\mathscr{B}_\lambda)$. By independency of the algebras $\{\mathscr{C}_\lambda\}_{\lambda\in\Lambda}$ we get

$$\mathbf{0}_{\mathscr{C}} \neq \bigwedge_{\lambda\in\Lambda_0} h_\lambda(b_\lambda)$$

in the algebra $\mathscr{C}$. But

$$\bigwedge_{\lambda\in\Lambda_0} h_\lambda(b_\lambda) \leqslant \varepsilon(k)h_{\mu_k}(a_k) = \varepsilon(k)x_k$$

for $k = 1, 2, \ldots, n$ and

$$\bigwedge_{\lambda\in\Lambda_0} h_\lambda(b_\lambda) \leqslant \varepsilon(k)h_{\nu_k}(a_k) = \varepsilon(k)y_k$$

for $k = 1, 2, \ldots, n$. It follows that

$$\mathbf{0}_{\mathscr{C}} \neq \bigwedge_{\lambda\in\Lambda_0} h_\lambda(b_\lambda) \leqslant \bigwedge_{k=1}^{n} \big(\varepsilon(k)x_k \wedge \varepsilon(k)y_k\big) \leqslant \bigwedge_{k=1}^{n} [\neg(x_k \Delta y_k)].$$

Hence

$$\bigvee_{k=1}^{n} (x_k \Delta y_k) \neq \mathbf{1}_{\mathscr{C}}.$$

Let $\|x\|$ denote the layer of the element x in $\mathscr{C}/_{\Delta}$ $(x \in C)$. We define a mapping $\psi : \mathscr{C}_0/_{\sim} \mapsto \mathscr{C}/_{\Delta}$ by the formula $\psi(|x|) = \|x\|$. We check that

ψ is well defined, i.e. if $|x| = |y|$, then $\|x\| = \|y\|$. Let $x \sim y$. Then $x \Delta y \in \Delta$. Hence $\|x\| = \|y\|$. We claim that ψ maps homomorphically $\mathscr{C}_0/_\sim$ into $\mathscr{C}/_\Delta$. It is obvious that $\psi(\neg|x|) = \neg(\psi(|x|))$ and that $\psi(|\mathbf{1}_\mathscr{C}|) = \|\mathbf{1}_\mathscr{C}\|$ is the unit element in $\mathscr{C}/_\Delta$. Suppose that $C_2'(|x|, |y|)$ in $\mathscr{C}_0/_\sim$. Hence $x = h_\mu(a)$, $y = h_\nu(b)$ and $C_2(a, b)$ in $\mathscr{B}$. There is a $\lambda_0 \in \Lambda$ such that $a, b \in B_{\lambda_0}$. Let $x_0 = h_{\lambda_0}(a)$ and $y_0 = h_{\lambda_0}(b)$. Then

$$\begin{aligned}\psi(|x| \vee |y|) &= \psi(|x_0| \vee |y_0|) = \psi(|x_0 \vee y_0|)\\ &= \|x_0 \vee y_0\| = \|x_0\| \vee \|y_0\| = \psi(|x|) \vee \psi(|y|).\end{aligned}$$

This completes the proof of Theorem 3.

In a similar manner one can prove the following theorems (see [2], p. 74):

THEOREM 4. *Let* $\mathscr{B} = \langle B; \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ *be a partial Boolean algebra in a broader sense. The following conditions are equivalent:*

(i) $\mathscr{B}$ *is weakly embeddable into a Boolean algebra.*

(ii) *For every finite sequence* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *of elements of* B *and any* a_{i_0} *belonging to* $(a_1, a_2, \ldots, a_n)$ $(a_{i_0} \neq \mathbf{0})$ *there exists a sequence* $(\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(n))$, *where* $\varepsilon(k) = -1$ *or* $\varepsilon(k) = +1$ $(k = 1, 2, \ldots, n)$, *such that*

1 $\varepsilon(i_0) = +1$;

2. *for every subsequence* $(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ *of* $\mathbf{a}$ *with the property* $C_m(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ *inequality* (9) *holds.*

THEOREM 5. *Let* $\mathscr{B} = \langle B; \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ *be a partial Boolean algebra in a broader sense. The following conditions are equivalent:*

(i) $\mathscr{B}$ *is embeddable into a Boolean algebra.*

(ii) *For every finite sequence* $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ *of elements of* B *and any two distinct elements* a_{i_1} *and* a_{i_2} $(a_{i_1} \neq a_{i_2})$ *belonging to* $(a_1, a_2, \ldots, a_n)$ *there exists a sequence* $(\varepsilon(1), \varepsilon(2), \ldots, \varepsilon(n))$, *where* $\varepsilon(k) = -1$ *or* $\varepsilon(k) = +1$ $(k = 1, 2, \ldots, n)$, *such that*

1. *either* $\varepsilon(i_1) = +1$, $\varepsilon(i_2) = -1$ *or* $\varepsilon(i_1) = -1$, $\varepsilon(i_2) = +1$;

2. *for every subsequence* $(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ *of* $\mathbf{a}$ *with the property* $C_m(a_{k_1}, a_{k_2}, \ldots, a_{k_m})$ *inequality* (9) *holds.*

Theorem 4 may also be formulated in terms of the direct limit of the system (6):

THEOREM 4*. *Suppose that* $\mathscr{B} = \langle B; \{C_n\}_{n\geqslant 2}, \vee, \neg, \mathbf{1}\rangle$ *is a partial Boolean algebra in a broader sense. Let* (6) *be the partially ordered system of Boolean algebras in* $\mathscr{B}$ *with property* C, *whose direct limit is* $(\mathscr{A}, \{i_\lambda : \lambda \in \Lambda\})$. *Then, for every* $\lambda \in \Lambda$, h_λ *is a monomorphism of* $\mathscr{B}_\lambda$ *into* $\mathscr{A}$ *iff* $\mathscr{B}$ *satisfies condition* (ii) *of Theorem* 4.

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