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LATTICE ORDERED GROUPS OF FINITE BREADTH

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The purpose of this note is to give the solution to a problem by Birkhoff (cf. [1], Problem 121) concerning lattice ordered groups (l-groups) of finite breadth (see Section 2). Let G be an l-group, and consider the following conditions on G:

(a) There are elements $x, y \in G$ such that $x \nless y$ and $2x < 2y$.

(b) There are elements $x, y \in G$ such that $x \neq y$ and $2x = 2y$.

There exist l-groups G satisfying (a) and (b) (cf. [1], p. 291, Example 5); Birkhoff asks if the pathological behaviour defined by conditions (a) and (b) can occur in an l-group of finite breadth.

We show that the answer is positive; for each positive integer $n > 1$ there is an l-group G such that the breadth of G equals n and G fulfils (a) and (b). It is not hard to verify that an l-group G has finite breadth n if and only if there exists a disjoint subset $S \subset G$ with $\operatorname{card} S = n$ and if no disjoint subset of G contains more than n elements. Such l-groups were studied by Conrad and Clifford [4] (for $n = 2$), Conrad [2], Kokorin and Hisamiev [6] and Kokorin and Kozlov [7]. In [2] (cf. also [3], where a more general situation was dealt with) it was proved that any such l-group G is a small lexicographic sum of linearly ordered groups A_i ($i = 1, \ldots, n$), where $\{A_i\}_{i=1}^{n} = \mathscr{S}$ is the system of all maximal linearly ordered subgroups of G. We show that, for a lattice ordered group of finite breadth, condition (a) is equivalent with any one of the following conditions:

(c) There exists $A_i \in \mathscr{S}$ such that A_i is not normal in G.

(d) There exists $0 < a \in G$ such that the interval $[0, a]$ is a chain and a is disjoint with some of its conjugates.

1. Preliminaries. For the standard concepts concerning lattices and lattice ordered groups cf. [1] and [5]. We recall the following notions (cf. [3]):

Let $G = (G; \wedge \vee, +)$ be an l-group, $a, b \in G$, $a \leqslant b$. The interval $[a, b]$ is the set $\{x \in G\colon a \leqslant x \leqslant b\}$. A subset $X \subset G$ is convex if $[x_1, x_2] \subset X$

whenever $x_1, x_2 \in X$ and $x_1 \leqslant x_2$. A subset $Y \subset G$ is called *disjoint* if $y > 0$ for each $y \in Y$ and $y_1 \wedge y_2 = 0$ for any pair of distinct elements $y_1, y_2 \in Y$. A system $\mathscr{S}$ of convex l-subgroups of G is said to be *disjoint* if for any two distinct l-subgroups $A_1, A_2 \in \mathscr{S}$ and any $a_1 \in A_1$, $a_2 \in A_2$ we have $|a_1| \wedge |a_2| = 0$. A disjoint system $\mathscr{S}$ is said to be *maximal* if it is not a proper subset of a disjoint system of convex l-subgroups of G. Let Y be a convex l-subgroup of G and $x \in G$. If $|x| \wedge |y| = 0$ for each $y \in Y$, we write $Y \delta x$.

Let $X_1, \ldots, X_n$ be convex l-subgroups of G such that the group $(G; +)$ is the direct sum of X_i $(i = 1, \ldots, n)$ and, for any $x_i \in X_i$, $x_1 + \ldots + x_n \geqslant 0$ if and only if $x_i \geqslant 0$ for $i = 1, \ldots, n$. Then the l-group G is said to be an *l-direct sum* of its l-subgroups X_i, and we write $G = X_1 \oplus \ldots \oplus X_n$.

More generally, let $\mathscr{T} = \{X_i\}$ $(i \in I)$ be a system of convex l-subgroups of G such that the group $(G; +)$ is the discrete direct sum of groups $(X_i; +)$ $(i \in I)$. Suppose that, for any finite subset $\{i_1, \ldots, i_n\} \subset I$ and any $x_{i_k} \in X_{i_k}$, the relation $x_{i_1} + \ldots + x_{i_n} \geqslant 0$ implies $x_{i_k} \geqslant 0$ $(k = 1, \ldots, n)$. Then G is the l-direct sum of the system $\mathscr{T}$ and we then write $G = \sum \oplus X_i$ $(i \in I)$. If $i_1, \ldots, i_n$ are distinct elements of I, $x_i \in X_i$ and $x = x_{i_1} + \ldots + x_{i_n}$, then we put $x_{i_k} = x(X_{i_k})$.

Now let I be a linearly ordered set and, for each $i \in I$, let X_i be an l-group such that X_i is linearly ordered whenever i is not the least element of I. Let H be the system of all mappings $f\colon I \to \bigcup X_i$ with $f(i) \in X_i$ for each $i \in I$. For $f \in H$ write $I(f) = \{i \in I\colon f(i) \neq 0\}$. Let G be the system of all $f \in H$ such that $I(f)$ is well ordered. We define in G the operation $+$ componentwise and we put $f > 0$ if $I(f) \neq \emptyset$ and $f(i_0) > 0$, where i_0 is the least element of $I(f)$. Then G is an l-group and it is called the *lexicographic product* of l-groups X_i; we denote it by $G = \Gamma X_i$ $(i \in I)$.

Let A be an l-ideal of G such that $g > a$ for any $a \in A$ and $g \in G^+ \setminus A$. Then G is a lexicographic extension of A and we then write $G = \langle A \rangle$. A lexicographic extension $G = \langle A \rangle$ is non-trivial if $G \neq A$.

Let $\mathscr{S}_1, \mathscr{S}_2, \ldots$ be systems of non-zero convex l-subgroups of G and $K = \{1, 2, \ldots\}$. For any $\mu \in K$ let A_μ be the convex l-subgroup of G that is generated by the set $\bigcup A_i^\mu$, where $\mathscr{S}_\mu = \{A_i^\mu\}$ $(i \in I_\mu)$. Assume that the following conditions are fulfilled:

(i) The system $\mathscr{S}_1$ is maximal disjoint.

(ii) If μ is a positive integer, $1 < \mu$ and $i \in I_\mu$, then either A_i^μ equals $A_{i_1}^{\mu-1}$ for some $i_1 \in I_{\mu-1}$ or there is a convex l-subgroup B of G and a finite subset $\{j_1, \ldots, j_m\} \subset I_{\mu-1}$, $m > 1$ such that $B = A_{j_1}^{\mu-1} \oplus \ldots \oplus A_{j_m}^{\mu-1}$ and A_i^μ is a non-trivial lexicographic extension of B.

(iii) $A_\mu = \Sigma \oplus A_i^\mu$ $(i \in I_\mu)$ and A_μ is an l-ideal of G for $\mu = 1, 2, \ldots$

(iv) $G = \bigcup A_\mu$ $(\mu \in K)$.

Then G is said to be a *small lexicographic sum* of l-groups of the system $\mathscr{S}_1$.

If elements $x, y \in G$ are incomparable, we write $x \mid y$.

2. Breadth of a lattice. Let L be a lattice. Suppose that $b = bL$ is the least positive integer such that any meet $x_1 \wedge \ldots \wedge x_n$ $(n > b)$ is always a meet of b of the x_i (cf. [1], p. 99). Then bL is the *breadth* of the lattice L; the lattice L is said to be *of finite breadth* if bL does exist.

A subset $X = \{x_1, \ldots, x_m\} \subset L$ will be called *irreducible* if $\inf(X \setminus \{x_i\}) > \inf X$ for each $i \in \{1, \ldots, m\}$. Then $bL = n > 1$ if and only if L contains an irreducible subset with n elements and if no subset $Y \subset L$ with $\operatorname{card} Y > n$ is irreducible.

Assertions 2.1-2.3 are easy to verify (cf. also [1], p. 32, Example 6).

2.1. *$bL = 1$ if and only if L is a chain.*

2.2. *If bL exists and L_1 is a sublattice of L, then bL_1 exists and $bL_1 \leqslant bL$.*

2.3. *If L is a direct product of lattices L_1 and L_2 with $\operatorname{card} L_i > 1$ $(i = 1, 2)$ and if bL_1 and bL_2 exists, then $bL = bL_1 + bL_2$.*

2.4. *Let X be a disjoint subset of an l-group G, $\operatorname{card} X = n$, and let H be the convex l-subgroup of G generated by X. Assume that each interval $[0, x]$ $(x \in X)$ is a chain. Then $bH = n$.*

Proof. Let $X = \{x_1, \ldots, x_n\}$. For any x_i there exists a maximal linearly ordered subgroup X_i of G containing x_i and the system $\{X_i\}$ $(i = 1, \ldots, n)$ is disjoint. Thus, from [2], Theorem 2, we infer that $H = X_1 \oplus \ldots \oplus X_n$; now, according to 2.1 and 2.3, we have $bH = n$, since the lattice H is isomorphic to a direct product of lattices X_i $(i = 1, \ldots, n)$.

Let $G = \langle H \rangle$ and $x_1, \ldots, x_n \in G$, $x_1 \wedge \ldots \wedge x_n = x$. Since G/H is linearly ordered, the set $\{x_1 + H, \ldots, x_n + H\}$ has the least element $x_k + H$ and $x_k < x_i$ for each $x_i \notin x_k + H$; because $x_k + H$ is a sublattice of G, we have $x \in x_k + H$ and $x = y_1 \wedge \ldots \wedge y_m$, where $\{y_1, \ldots, y_m\} = \{x_i\colon x_i \in x_k + H\}$.

2.5. *Let $G = \langle H \rangle$, $bH = n$. Then $bG = n$.*

Proof. If bG does exist, then $bG \geqslant n$ by 2.2. Let $m \geqslant n$, $x_i \in G$ $(i = 1, \ldots, m)$, $x_1 \wedge x_2 \wedge \ldots \wedge x_m = x$ and assume that $X = \{x_1, \ldots, x_m\}$ is an irreducible subset of G. There is a subset $Y \subset X$ such that $Y \subset H + x$ and $\inf Y = \inf X$. Since X is irreducible, we have $X = Y$. Write $x_i - x = z_i$. Then $\{z_1, \ldots, z_m\}$ is an irreducible subset of H, hence $m \leqslant n$. This shows that $bG = n$.

2.6. *If G is a small lexicographic sum of a system $\mathscr{S}_1 = \{B_1, \ldots, B_n\}$ of non-zero linearly ordered groups, then $bG = n$.*

Proof. Assume that G is a small lexicographic sum of a finite system $\mathscr{S}_1 = \{B_1, \ldots, B_n\}$ of non-zero linearly ordered groups. Then there is a positive integer $k \leqslant n$ such that (the notation is as in Section 1) $G = A_k$. From 2.1, 2.3 and 2.5 we infer (by induction) that $bA_m = n$ for $m = 1, \ldots, k$. Thus $bG = n$.

Consider the following example (cf. [1], p. 216, Example 6):

Let N be the set of all integers and let G be the set of all triples (x, y, z) with $x, y, z \in N$. We define the operation $+$ in G by the rule

$$(x_1, y_1, z_1)+(x_2, y_2, z_2) = (x_1+x_2, y_3, z_3),$$

where $y_3 = y_1+y_2$ and $z_3 = z_1+z_2$ if x_2 is even, $y_3 = z_1+y_2$ and $z_3 = y_1+ +z_2$ if x_2 is odd.

Further, we put $(x, y, z) \geqslant 0$ if either $x > 0$ or $x = 0$ and $y \geqslant 0$, $z \geqslant 0$. Then G is a lattice ordered group. Write $Y = \{(0, y, 0): y \in N\}$ and $Z = \{(0, 0, z): z \in N\}$. The l-group G is a small lexicographic sum of l-groups Y, Z and these are linearly ordered. Therefore, according to 2.6, we have $bG = 2$.

Put $a = (1, 0, 0)$ and $b = (1, 1, -1)$. Then $a \neq b$ and $2a = (2, 0, 0) = 2b$, thus G fulfils (b). Further, write $c = (1, 2, -1)$. The elements a, c are incomparable and $2c = (2, 1, 1) > 2a$, hence G satisfies (a). Let n be a positive integer, $n-2 = k > 0$. Let B be the direct sum of k copies of N, $H = G \oplus B$. Then, $bH = n$ and H fulfils (a) and (b). Therefore, we have

2.7. *For any positive integer $n \geqslant 2$ there is a lattice ordered group G fulfilling* (a) *and* (b) *such that $bG = n$.*

2.8. *Let G be an l-group. The following conditions are equivalent*:

(i) $bG = n$;

(ii) *G contains a disjoint subset with n elements and it does not contain any disjoint subset with more than n elements.*

Proof. Assume that (i) holds and let $X = \{x_1, \ldots, x_m\}$ be a disjoint set. Let H be the l-subgroup of G generated by X. Then $H \cap [0, x_i] = \{0, x_i\}$. According to 2.4, we have $bH = m$, whence $bG \geqslant m$, and this implies $m \leqslant n$. Thus there exists a disjoint subset X of G with the greatest cardinality $m_0 \leqslant n$. From [2], Theorem 1, it follows that G is a small lexicographic sum of m_0 non-zero linearly ordered groups and hence, by 2.6, $bG = m_0$. Thus $n = m_0$, and so (ii) is satisfied. Conversely, let (ii) hold. By [2] and 2.6, we obtain $bG = n$.

3. Condition (F). Let us consider the following condition on $G \neq \{0\}$:

(F) Any bounded disjoint subset of G is finite.

3.1 (cf. [3], Theorem 6.1). *G fulfils* (F) *if and only if it is a small lexicographic sum of linearly ordered groups.*

From the proof of this theorem that may be found in [3] it follows that if G satisfies (F), then it is a small lexicographic sum of the system

$\mathcal{S}_1$ consisting of all maximal non-zero linearly ordered subgroups of G. According to 2.8, any l-group G of finite breadth satisfies (F). If G satisfies (F), then it need not be of finite breadth.

Assume that G satisfies (F). Let $\mathcal{S}_1$ be as above and let $\mathcal{S}_n$ $(n = 2, 3, \ldots)$ be as in Section 1.

3.2. *Assume that each l-group A_i^1 $(i \in I_1)$ is an l-ideal of G. Then, for each $n > 1$ and each $i \in I_n$, the l-group A_i^n is an l-ideal of G.*

Proof. Assume that the assertion is valid for $n-1$, where $n > 1$, and let $i \in I_n$. There is $i_1 \in I_{n-1}$ such that $A_{i_1}^{n-1} \subset A_i^n$. Thus, for any $x \in G$, we have $A_{i_1}^{n-1} = x + A_{i_1}^{n-1} - x \subset x + A_i^n - x$. Moreover, from the construction of $\mathcal{S}_n$ devised in [3], it follows that there is $i_2 \in I_n$ such that $x + A_i^n - -x = A_{i_2}^n$ and any two distinct elements of $\mathcal{S}_n$ are disjoint. Since $A_{i_1}^{n-1} \subset A_i^n \cap A_{i_2}^n$, we obtain $A_{i_2}^n = A_i^n$.

3.3. *Assume that there is $A_{i_0}^1 \in \mathcal{S}_1$ such that $A_{i_0}^1$ is not an l-ideal of G and the number of l-groups that are conjugate to $A_{i_0}^1$ is finite. Then G fulfils* (a).

Proof. According to the assumption, there is $x \in G$ such that $-x + +A_{i_0}^1 + x \neq A_{i_0}^1$. Consider the mapping φ: $g \to -x + g + x$ $(g \in G)$. Each of the l-groups

$$\varphi(A_{i_0}^1), \varphi^2(A_{i_0}^1), \ldots, \varphi^n(A_{i_0}^1), \ldots \tag{1}$$

is conjugate to $A_{i_0}^1$, hence the sequence (1) is finite and since φ is an automorphism on G, there is the least positive integer $n > 1$ such that $\varphi^n(A_{i_0}^1) = A_{i_0}^1$. Then $A_{i_0}^1$, $\varphi(A_{i_0}^1), \ldots, \varphi^{n-1}(A_{i_0}^1)$ are distinct l-groups. Choose $0 < z \in A_{i_0}^1$ and write

$$y = x + z - \varphi(z) + \varphi^2(z) - \varphi^3(z) + \ldots + (-1)^{n-1}\varphi^{n-1}(z), \qquad y_1 = -x + y.$$

Since φ^k $(k = 1, 2, \ldots)$ is an automorphism on the l-group G, each l-group $\varphi^k(A_{i_0}^1)$ is a maximal linearly ordered subgroup of G, hence belongs to $\mathcal{S}_1$. Therefore, the system

$$\mathcal{S}_0 = \{A_{i_0}^1, \varphi(A_{i_0}^1), \ldots, \varphi^{n-1}(A_{i_0}^1)\}$$

is disjoint and so, according to Theorem 2 of [2], the convex l-subgroup H of G generated by the subgroups belonging to $\mathcal{S}_0$ is the l-direct sum of l-groups $A_{i_0}^1, \ldots, \varphi^{n-1}(A_{i_0}^1)$. We have $y_1 \in A_{i_0}^1 \oplus \varphi(A_{i_0}^1) \oplus \ldots \oplus \varphi^{n-1}(A_{i_0}^1)$ and $\varphi^k(z) > 0$ $(k = 0, \ldots, n-1)$; thus the element y_1 is incomparable with 0 and, therefore, $y \mid x$. Further, we have

$$\varphi(y_1) = \varphi(z) - \varphi^2(z) + \varphi^3(z) - \ldots + (-1)^{n-1}\varphi^n(z),$$

whence

$$2y = 2x + \varphi(y_1) + y_1 = 2x + (-1)^{n-1}\varphi^n(z) + z.$$

Now, we distinguish two cases:

(i) Suppose that $(-1)^{n-1}\varphi^n(z)+z \neq 0$. Since $\varphi^n(A^1_{i_0}) = A^1_{i_0}$, we have $(-1)^{n-1}\varphi^n(z)+z \in A^1_{i_0}$ and because $A^1_{i_0}$ is linearly ordered, we infer that the elements $2x$ and $2y$ are comparable and distinct; therefore, (a) is valid.

(ii) Assume that $(-1)^{n-1}\varphi^n(z)+z = 0$. Write $y_2 = z+y_1$, $y' = x+y_2$. Then we get $y' \mid x$, and

$$\begin{aligned}2y' &= 2x+\varphi(y_2)+y_2 = 2x+\varphi(z)+\varphi(y_1)+z+y_1\\ &= 2x+\varphi(z)+\big((-1)^{n-1}\varphi^n(z)+z\big)+z = 2x+\varphi(z)+z > 2x,\end{aligned}$$

whence (a) holds.

As a corollary to 3.3, 2.8, and 3.1 we obtain

3.4. *Let G be an l-group of finite breadth and assume that there exists $A^1_i \in \mathscr{S}_1$ such that A^1_i is not normal in G. Then G fulfils* (a).

3.5. *Assume that G satisfies* (F) *and that each $A^1_i (i \in I_1)$ is normal. For $0 < x \in G$, let $I(x) = \{i \in I_1\colon 0 < a \leqslant x$ for some $a \in A^1_i\}$. Then, for $x, y \in G^+$, we have $x \wedge y = 0$ if and only if $I(x) \cap I(y) = \emptyset$.*

Proof. Let $x \wedge y = 0$. Assume that $i \in I(x) \cap I(y)$. There exist $a_1, a_2 \in A^1_i$ with $0 < a_1 \leqslant x$, $0 < a_2 \leqslant y$; because A^1_i is linearly ordered, we have $0 < a_1 \wedge a_2 \in A^1_i$ and $a_1 \wedge a_2 \leqslant x \wedge y = 0$, a contradiction. Conversely, let $I_1(x) \cap I_1(y) = \emptyset$. If $x \wedge y = z > 0$, then (since the system $\mathscr{S}_1$ is maximal disjoint) there is $i \in I_1$ and $0 < a \in A^1_i$ with $a \leqslant z$. From this we obtain $i \in I_1(x) \cap I_1(y)$, a contradiction.

Let φ have the same meaning as in 3.3.

3.6. *Assume that G satisfies* (F) *and that each $A^1_i (i \in I_1)$ is normal. Let $a, b \in G$, $a \wedge b = 0$. Then $a \wedge \varphi(b) = 0$.*

Proof. Suppose to the contrary that $a \wedge \varphi(b) = c > 0$. Then $I(c) \neq \emptyset$; let $i \in I(c)$. Thus $i \in I(a)$ and $i \in I(\varphi(b))$. Therefore, A^1_i non $\delta\varphi(b)$, whence $\varphi^{-1}(A^1_i)$ non δb. But $\varphi^{-1}(A^1_i) = A^1_i$ and so A^1_i non δb, and this implies $i \in I(b)$. We get $i \in I(a) \cap I(b)$; thus, according to 3.5, $a \wedge b \neq 0$, a contradiction.

3.7. *Assume that G satisfies* (F) *and each $A^1_i \in \mathscr{S}_1$ is normal. Then G fulfils neither* (a) *nor* (b).

Proof. Suppose that there are elements $x, y \in G$ with $x \mid y$, $2x \leqslant 2y$. There is a positive integer n such that $x, y, 2x, 2y \in A_n$; let n be the least positive integer with this property. Since A_n is discrete direct sum of l-groups A^n_i $(i \in I_n)$, there must exist $i_0 \in I_n$ such that $x(A^n_{i_0}) \mid y(A^n_{i_0})$, $2x(A^n_{i_0}) = 2y(A^n_{i_0})$. Write $x(A^n_{i_0}) = x_1$, $y(A^n_{i_0}) = y_1$. It cannot occur that $A^n_{i_0} \leqslant A^{n-1}_i$ for some $i \in I_{n-1}$ because of the minimality of n; moreover, $n > 1$ since the l-groups A^1_i are linearly ordered. Thus $A^n_{i_0} = \langle B\rangle$, $A^n_{i_0} \neq B$ and B is a direct sum of two or more l-groups belonging to $\mathscr{S}_{n-1}$. From $x_1 \mid y_1$ we obtain $x_1+B = y_1+B$, whence $z = -x_1+y_1 \in B$, $z \mid 0$. Therefore, $z = z^+ - z^-$, $z^+ > 0$, $z^- > 0$ and $z^+ \wedge z^- = 0$. Hence, $y_1 = x_1 + z$ and $2y_1 = 2x_1 + \varphi(z^+) - \varphi(z^-) + z^+ - z^-$.

According to 3.6, we have $\varphi(z^+) \wedge z^- = 0$, $\varphi(z^-) \wedge z^+ = 0$ and, clearly, $\varphi(z^+) \wedge \varphi(z^-) = 0$; therefore, $(\varphi(z^+) + z^+) \wedge (\varphi(z^-) + z^-) = 0$. From this it follows that $\varphi(z^+) - \varphi(z^-) + z^+ - z^- \mid 0$, whence $2y_1 \mid 2x_1$, a contradiction.

3.8. THEOREM. *Assume that G satisfies* (F) *and that each $A_i^1 \in \mathscr{S}_1$ has only a finite number of conjugates. Then condition* (a) *is equivalent with any one of the following conditions*: (i) *there exists $A_i^1 \in \mathscr{S}_1$ that is not normal in G*; (ii) *there exists $0 < a \in G$ such that $[0, a]$ is a chain and the element a is disjoint with some of its conjugates. Moreover, if G satisfies* (b), *then it fulfils* (a) *as well.*

Proof. The equivalence of conditions (a) and (i) follows from 3.3 and 3.7. Assume that (i) is valid and choose $0 < a \in A_i^1$. There is $x \in G$ such that $-x + A_i^1 + x = A_{i_1}^1 \in \mathscr{S}_1$, $A_i^1 \neq A_{i_1}^1$, hence $A_i^1 \cap A_{i_1}^1 = \{0\}$ and, therefore, $a \wedge (-x + a + x) = 0$. Since A_i^1 is linearly ordered, $[0, a]$ is a chain. Conversely, let (ii) be satisfied. Because $[0, a]$ is a chain, there is $A_i^1 \in \mathscr{S}_1$ with $a \in A_1^1$. If $a \wedge (-x + a + x) = 0$ for some $x \in G$, then $-x + a + x \notin A_i^1$, hence $-x + A_i^1 + x \neq A_i^1$. If (b) is valid, then, according to 3.7, (i) holds, and so (a) is satisfied.

As a corollary we obtain

3.8.1. *Let G be an l-group of finite breadth. Then conditions* (a), (i) *and* (ii) *from* 3.8 *are equivalent. If G satisfies* (b), *then it fulfils* (a) *as well.*

3.9. *There exist l-groups of finite breadth satisfying* (a) *and not fulfilling* (b).

Example. Let I be the set of all integers with the natural order and let $X_i = N = I$ for each $i \in I$, $X = Y = \Gamma_{i \in I} X_i$. Let G be the set of all triples (n, x, y) with $n \in N$, $x \in X$, $y \in Y$. For any $n \in N$ and $x \in X$ let $x^n \in X$ such that $x^n(i) = x(i+n)$. Define the operation $+$ in G by the rule

$$(n_1, x_1, y_1) + (n_2, x_2, y_2) = (n_1 + n_2, x_3, y_3),$$

where $x_3 = x_1^{n_2} + x_2$, $y_3 = y_1^{n_2} + y_2$ for n_2 even, and $x_3 = y_1^{n_2} + x_2$, $y_3 = x_1^{n_2} + y_2$ for n_2 odd. $(G; +)$ is a group. Put $(n_1, x_1, y_1) \geqslant 0$ if either $n_1 > 0$ or $n_1 = 0$ and $x_1 \geqslant 0$, $y_1 \geqslant 0$. Then G is an l-group, $bG = 2$, and G fulfils (a) but not (b).

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