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FUNCTION $\cup$-SEMIGROUPS

BY

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A *function* in a set A is any (in particular, empty) partial transformation of A. If f and g are functions in A, then $g\circ f$ denotes a function in A such that, for every $a\in A$, $g\circ f(a)$ is defined if and only if $f(a)$ and $g(f(a))$ are defined and $g\circ f(a) = g(f(a))$. A *function semigroup* is any non-empty set of functions in a fixed set A closed under superposition $\circ$ of functions. Clearly, $\circ$ is an associative operation.

We consider functions in A as binary relations

$$f = \{(a, f(a)) \colon a \in \mathrm{pr}_1 f\} \subset A \times A,$$

where $\mathrm{pr}_1 f$ is the domain of f. Thus, set-theoretic operations on functions make sense; e.g., $f\cup g$ denotes the set-theoretical union of f and g. Obviously, $f \cup g$ need not be a function. If $f\cup g$ is a function, then f and g are called *compatible*. A function semigroup is called *compatible* if every two functions belonging to the semigroup are compatible. Compatible function semigroups have been studied in our previous paper [11].

Every function semigroup is (partially) ordered by the inclusion relation: $f \subset g$ means that g is an extension of f. This order relation has been studied in another paper of the author [4] (cf. also [6] and [8]). In particular, linearly inclusion-ordered function semigroups have been considered in [9]. Clearly, a linearly inclusion-ordered function semigroup is compatible. In this paper we consider other types of inclusion-orderings on compatible semigroups.

In Theorems 1 and 2, necessary and sufficient conditions are given for an ordered semigroup to be order-isomorphic with an inclusion-ordered compatible or strongly compatible function semigroup. (A function semigroup F is called *strongly compatible* if all functions from F are one-to-one and the set-theoretic union of any two functions from F is one-to-one.)

Suppose F is a function semigroup closed under the binary set-theoretic union $\cup$. Then F (considered as an algebra with respect to two binary operations: superposition and union of functions) is called a *function $\cup$-semigroup. Function $\cap$-semigroups* are defined analogously. Clearly, function $\cup$-semigroups are compatible as semigroups. Function $\cap$-semigroups need not be compatible (e.g., the symmetric function semigroup $\mathcal{F}_A$ of all functions in A is closed under $\cap$). Function $\cup$-semigroups are semilattice-ordered. In Theorems 3 and 4 we characterize those semilattice-ordered semigroups which are order-isomorphic with function $\cup$-semigroups and one-to-one function $\cup$-semigroups. An analogous problem for function $\cap$-semigroups has been solved by Garvackiĭ [1]. At the end of this paper we give some results on the structure of semilattice-ordered semigroups order-isomorphic with function $\cup$-semigroups.

THEOREM 1. *An ordered semigroup $(S; \cdot, \leqslant)$ (here $(S; \cdot)$ is a semigroup and $(S; \leqslant)$ an ordered set) is order-isomorphic with an inclusion-ordered compatible function semigroup if and only if the order $\leqslant$ is stable, i.e., for every $s, t, u, v \in S$,*

(1) $\quad s \leqslant t$ *and* $u \leqslant v$ *imply* $su \leqslant tv$,

and, for every $s, t, u \in S$, the following two conditions are satisfied:

(2) $\quad$ *if* $s \leqslant tu$, *then* $s \leqslant su$,

(3) $\quad st \leqslant s$.

Proof. Necessity. Suppose $(F; \circ, \subset)$ is an inclusion-ordered function semigroup. The necessity of (1) is well known [4]. Now suppose $f \subset h \circ g$ for some $f, g, h \in F$. If $f(a)$ is defined, then $h(g(a))$ is defined. Therefore, $g(a)$ is defined. Since F is compatible, $f(a) = g(a)$, whence $f(a) = h(g(a)) = h(f(a))$ and $f \subset h \circ f$. Therefore, (2) is necessary. Notice that in a function semigroup the product of two factors f and g is written from right to left: $g \circ f$; in an abstract semigroup the same product is written from left to right: fg.

Clearly, $\mathrm{pr}_1(g \circ f) \subset \mathrm{pr}_1 f$ (cf., for example, [4]). Since $g \circ f$ and f are compatible, $g \circ f \subset f$ and (3) is necessary.

Sufficiency. Suppose (1)-(3) hold. By (3), $xy^2 \leqslant xy$; by (2), $xy \leqslant xy$ implies $xy \leqslant xy^2$; therefore, $xy^2 = xy$ for all $x, y \in S$. By this identity, and by (3) and (1), $xy \leqslant x$ implies $xyz = x(yz)^2 = xyzyz \leqslant xzyz \leqslant xzy$; analogously, $xzy \leqslant xyz$. Therefore, $xyz = xzy$, i.e., the semigroup $(S; \cdot)$ satisfies identities (1) and (2) of Theorem 1 from [11]. The following part of the proof is an almost exact replica of the proof of the sufficiency of Theorem 1 from [11], the only difference being that $\mathsf{P}(g)(a_x)$ is defined only if $x \leqslant g$. As in [11], we obtain $\mathsf{P}(gh) = \mathsf{P}(h) \circ \mathsf{P}(g)$ for all $g, h \in S$.

If $\mathsf{P}(g) \subset \mathsf{P}(h)$, then $\mathsf{P}(g)(a_g)$ is defined; therefore, $\mathsf{P}(h)(a_g)$ is defined, i.e., $g \leqslant h$. Conversely, if $g \leqslant h$ and $\mathsf{P}(g)(a_x)$ is defined, i.e., $x \leqslant g$, then $x \leqslant h$ and $\mathsf{P}(h)(a_x)$ is defined; if $\mathsf{P}(g)(b_x)$ is defined, then $xg = x$ and, by (1) and (3), $x = xg \leqslant xh \leqslant x$ and $xh = x$, i.e., $\mathsf{P}(h)(b_x)$ is defined.

Therefore, $\mathrm{pr}_1 \mathsf{P}(g) \subset \mathrm{pr}_1 \mathsf{P}(h)$ and, since the functions $\mathsf{P}(g)$ and $\mathsf{P}(h)$ are compatible, $\mathsf{P}(g) \subset \mathsf{P}(h)$. Thus,

$$\mathsf{P}(g) \subset \mathsf{P}(h) \leftrightarrow g \leqslant h.$$

In particular, if $\mathsf{P}(g) = \mathsf{P}(h)$, then $\mathsf{P}(g) \subset \mathsf{P}(h) \subset \mathsf{P}(g)$ and $g \leqslant h \leqslant g$, i.e., $g = h$. Therefore, P is an order-isomorphism of $(S; \cdot, \leqslant)$ onto an inclusion-ordered compatible function semigroup.

THEOREM 2. *An ordered semigroup $(S; \cdot, \leqslant)$ is order-isomorphic with an inclusion-ordered strongly compatible function semigroup if and only if conditions* (1)-(3) *of Theorem* 1 *hold and the semigroup $(S; \cdot)$ is commutative.*

Proof. Necessity. By Theorem 1, conditions (1)-(3) are necessary, and by Theorem 2 of [11], commutativity is necessary.

Sufficiency. As in the proof of the sufficiency of Theorem 2 from [11], suppose $a_x = b_x$ if and only if $x^2 = x$ and define $\mathsf{P}(g)$ for all $g \in S$ as in [11]. The only difference is that $\mathsf{P}(g)(a_x)$ is defined only if $x \leqslant g$. If $a_x = b_x$, then $x^2 = x$ and $x \leqslant g$ implies, by (1) and (3), $x = x^2 \leqslant xg \leqslant x$, i.e., $xg = x$. Conversely, $xg = x$ implies, by (3), $x = gx \leqslant g$. Therefore, $\mathsf{P}(g)(a_x)$ is defined if and only if $\mathsf{P}(g)(b_x)$ is. The next part of the proof is an obvious modification of the proof of the sufficiency of Theorem 2 from [11]. In particular, $\mathsf{P}(gh) = \mathsf{P}(h) \circ \mathsf{P}(g)$ for all $g, h \in S$ and the functions $\mathsf{P}(g)$ are strongly compatible for all $g \in G$. An obvious modification of the proof of Theorem 1 gives us the equivalence $\mathsf{P}(g) \subset \mathsf{P}(h) \leftrightarrow g \leqslant h$ showing that P is an order-isomorphism of $(S; \cdot, \leqslant)$ onto an inclusion-ordered strongly compatible function semigroup.

THEOREM 3. *A semilattice-ordered semigroup $(S; \cdot, \vee)$ is isomorphic with a function $\cup$-semigroup (i.e., there exists a one-to-one mapping P of S onto a set of functions such that $\mathsf{P}(t) \circ \mathsf{P}(s) = \mathsf{P}(st)$ and $\mathsf{P}(s) \cup \mathsf{P}(t) = \mathsf{P}(s \vee t)$ for all $s, t \in S$) if and only if the multiplication $\cdot$ is distributive relatively to the semilattice join $\vee$, i.e.*

(4) $\quad x(y \vee z) = xy \vee xz$ *and* $(y \vee z)x = yx \vee zx,$

and, if $x \leqslant y$ means that $x \vee y = y$, condition (3) *is satisfied and*

(5) $\quad s \leqslant t \vee uv$ *implies* $s \leqslant t \vee sv$ *for all* $s, t, u, v \in S.$

Proof. Necessity. Suppose $(F; \circ, \cup)$ is a function $\cup$-semigroup. Then identities (4) hold obviously and (3) follows from Theorem 1. Suppose $s \subset t \cup (v \circ u)$ for some $s, t, u, v \in F$. If $s(a)$ is defined, then either $t(a)$ or $v \circ u(a)$ (or both) are defined. If $v \circ u(a)$ is defined, then $u(a)$ is defined and, since s and u are compatible, $s(a) = u(a)$, i.e., $v \circ s(a)$ is defined. Thus if $s(a)$ is defined, then either $t(a)$ or $v \circ s(a)$ are defined, i.e., $\mathrm{pr}_1 s \subset \mathrm{pr}_1(t \vee v \circ s)$. Since the functions s and $t \vee v \circ s$ are compatible, $s \subset t \vee (v \circ s)$, i.e., (5) holds.

S u f f i c i e n c y. Suppose $(S; \cdot, \vee)$ is a semilattice-ordered semigroup satisfying conditions (3), (5) and (4).

An *order-ideal* (*o-ideal* for short) of $(S; \cdot, \vee)$ is a non-empty subset W of S satisfying the following two conditions: (i) if $s \leqslant t$ and $t \in W$, then $s \in W$; (ii) if $s, t \in W$, then $s \vee t \in W$.

Suppose g and h are different elements of S. Then at least one of two inequalities, $g \leqslant h$ and $h \leqslant g$, does not hold. Without loss of generality we can suppose that $g \leqslant h$ does not hold. Then the set $\{s\colon s \leqslant h\}$ is an o-ideal containing h but not g. By the Zorn lemma, there exists a maximal o-ideal W such that $h \in W$ and $g \in W'$, where $W' = S \setminus W$. Suppose that $s \in S$. Then, by (3), $ws \leqslant w$ for every $w \in W$ and $ws \in W$, i.e., $Ws \subset W$.

If $gs \in W$ and $us \in W'$, then the set $\{v\colon v \leqslant w \vee us$ for some $w \in W\}$ is an o-ideal of S which properly contains W. Therefore, $g \leqslant w \vee us$ for some $w \in W$. By (5), $g \leqslant w \vee gs$ and, since $w, gs \in W$, we infer that $g \in W$ — a contradiction. Therefore, $us \in W$, i.e., $Ss \subset W$.

If $gs \in W'$, $v \in W'$ and $vs \in W$, then the set $\{u\colon u \leqslant w \vee v$ for some $w \in W\}$ is an o-ideal which properly contains W, and therefore, $g \leqslant w \vee v$ for some $w \in W$. Using (1) (which is an obvious corollary to (4)), (4) and (3), we obtain

$$gs \leqslant (w \vee v)s = ws \vee vs \leqslant w \vee vs$$

and, since $w, vs \in W$, we infer that $gs \in W$ — a contradiction. Therefore, $vs \in W'$ and $W's \subset W'$.

An o-ideal V is called a *face* if, for every $s \in S$, the subset $V's$ is included in V' or in V (i.e., V is a *face* if V is an o-ideal and $(V \times V) \cup (V' \times V')$ is a right regular equivalence relation on $(S; \cdot)$). In particular, the o-ideal W considered above is a face containing h but not g. Thus we have proved that if $g \in V \leftrightarrow h \in V$ for all faces V, then $g = h$.

Now let $\boldsymbol{F}$ be the set of all faces. For every face $f \in \boldsymbol{F}$, let a_f and b_f be two symbols; if $f, g \in \boldsymbol{F}$ are two different faces, then all four symbols a_f, b_f, a_g and b_g are different. Let A be the set of all these symbols. For every $s \in S$, let $\mathsf{P}(s)$ be a function in A defined as follows:

$$a_x \in \mathrm{pr}_1 \mathsf{P}(s) \leftrightarrow s \in x'; \qquad b_x \in \mathrm{pr}_1 \mathsf{P}(s) \leftrightarrow x's \subset x';$$

if $a_x \in \mathrm{pr}_1 \mathsf{P}(s)$, then $\mathsf{P}(s)(a_x) = b_x$; if $b_x \in \mathrm{pr}_1 \mathsf{P}(s)$, then $\mathsf{P}(s)(b_x) = b_x$.

Clearly, $\mathsf{P}(s)$ is a function and all the functions $\mathsf{P}(s)$ for $s \in S$ are compatible. We are going to prove that P is an isomorphism of $(S; \cdot, \vee)$ onto a function semigroup.

Suppose $s, t \in S$ and $a_x \in \mathrm{pr}_1(\mathsf{P}(s) \cup \mathsf{P}(t))$, i.e., $a_x \in \mathrm{pr}_1 \mathsf{P}(s)$ or $a_x \in \mathrm{pr}_1 \mathsf{P}(t)$, i.e., $s \in x'$ or $t \in x'$. It follows that $s \vee t \in x'$. Conversely, if $s \vee t \in x'$, then $s \in x'$ or $t \in x'$ (since $s, t \in x$ imply $s \vee t \in x$). Now $s \vee t \in x'$ means that $a_x \in \mathrm{pr}_1 \mathsf{P}(s \vee t)$. If $b_x \in \mathrm{pr}_1(\mathsf{P}(s) \cup \mathsf{P}(t))$, i.e., $b_x \in \mathrm{pr}_1 \mathsf{P}(s)$ or $b_x \in \mathrm{pr}_1 \mathsf{P}(t)$, or, which is the same, $x's \subset x'$ or $x't \subset x'$, then the inequalities $s \leqslant s \vee t$

and $t \leqslant s \vee t$ imply the impossibility of $x'(s \vee t) \subset x$, i.e., $x'(s \vee t) \subset x'$. On the other hand, if $x'(s \vee t) \subset x'$, then $x's \subset x'$ or $x't \subset x'$ (otherwise, $x's \subset x$ and $x't \subset x$, whence $x'(s \vee t) \subset x$, since x is an o-ideal). Thus $b_x \in \mathrm{pr}_1 \mathsf{P}(s \vee t)$. We have proved that

$$\mathrm{pr}_1\big(\mathsf{P}(s) \cup \mathsf{P}(t)\big) = \mathrm{pr}_1 \mathsf{P}(s \vee t)$$

and, since the functions $\mathsf{P}(s) \cup \mathsf{P}(t)$ and $\mathsf{P}(s \vee t)$ are compatible,

$$\mathsf{P}(s) \cup \mathsf{P}(t) = \mathsf{P}(s \vee t).$$

Let $a_x \in \mathrm{pr}_1 \mathsf{P}(st)$, i.e., $st \in x'$. Since, by (3), $st \leqslant s$, we infer that $s \in x'$ and $a_x \in \mathrm{pr}_1 \mathsf{P}(s)$. Since $s \in x'$ and $st \in x'$, the inclusion $x't \subset x$ is impossible; therefore, $x't \subset x'$ and $b_x \in \mathrm{pr}_1 \mathsf{P}(t)$. It follows that $a_x \in \mathrm{pr}_1\big(\mathsf{P}(t) \circ \mathsf{P}(s)\big)$. Conversely, $a_x \in \mathrm{pr}_1\big(\mathsf{P}(t) \circ \mathsf{P}(s)\big)$ means that $a_x \in \mathrm{pr}_1 \mathsf{P}(s)$ and $b_x \in \mathrm{pr}_1 \mathsf{P}(t)$, i.e., that $s \in x'$ and $x't \subset x'$. It follows that $st \in x'$ and $a_x \in \mathrm{pr}_1 \mathsf{P}(st)$. Now $b_x \in \mathrm{pr}_1 \mathsf{P}(st)$ means that $x'st \subset x'$. If $u \in x'$, then, by (3), $ust \leqslant us$ and $us \in x'$, i.e., $x's \subset x'$. Now $x'st \subset x't$ and, since $x'st \subset x'$, we obtain $x't \subset x'$. Thus $b_x \in \mathrm{pr}_1 \mathsf{P}(s)$ and $b_x \in \mathrm{pr}_1 \mathsf{P}(t)$. It follows that $b_x \in \mathrm{pr}_1\big(\mathsf{P}(t) \circ \mathsf{P}(s)\big)$. Conversely, suppose $b_x \in \mathrm{pr}_1\big(\mathsf{P}(t) \circ \mathsf{P}(s)\big)$ or, equivalently, $b_x \in \mathrm{pr}_1 \mathsf{P}(s)$ and $b_x \in \mathrm{pr}_1 \mathsf{P}(t)$. Then $x's \subset x'$ and $x't \subset x'$, whence, $x'st \subset x't \subset x'$ and $b_x \in \mathrm{pr}_1 \mathsf{P}(st)$. Thus

$$\mathrm{pr}_1\big(\mathsf{P}(t) \circ \mathsf{P}(s)\big) = \mathrm{pr}_1 \mathsf{P}(st)$$

and, since the functions $\mathsf{P}(t) \circ \mathsf{P}(s)$ and $\mathsf{P}(st)$ are obviously compatible,

$$\mathsf{P}(t) \circ \mathsf{P}(s) = \mathsf{P}(st) \quad \text{for all } s, t \in S.$$

Now suppose $\mathsf{P}(s) = \mathsf{P}(t)$. If $x \in F$, then

$$s \in x' \leftrightarrow a_x \in \mathrm{pr}_1 \mathsf{P}(s) \leftrightarrow a_x \in \mathrm{pr}_1 \mathsf{P}(t) \leftrightarrow t \in x',$$

whence $s \in x \leftrightarrow t \in x$ for all $x \in F$. As we have proved above, $s = t$ follows. The proof of Theorem 3 is complete.

THEOREM 4. *A semilattice-ordered semigroup* $(S; \cdot, \vee)$ *is isomorphic with a* $\cup$*-semigroup of one-to-one functions if and only if conditions* (3), (4) *and* (5) *are satisfied and the semigroup* $(S; \cdot)$ *is commutative.*

Proof. The necessity follows from Theorem 3 and from Theorem 2 of paper [11].

Sufficiency. Introduce a_x and b_x for every $x \in F$ as in the proof of the sufficiency of Theorem 3 but with $a_x = b_x$ if and only if $x'x' \subset x'$. Let $\mathsf{P}(s)$ for every $s \in S$ be given as in the proof of Theorem 3. Suppose $a_x = b_x$. It is a matter of a simple evaluation to check (using commutativity of $(S; \cdot)$) that

$$a_x \in \mathrm{pr}_1 \mathsf{P}(s) \leftrightarrow b_x \in \mathrm{pr}_1 \mathsf{P}(s) \quad \text{for every } s \in S.$$

Clearly, $\mathsf{P}(s)$, $s \in S$, form a compatible set of functions. To prove that $\mathsf{P}(s)$ is one-to-one we need to consider the case where $\mathsf{P}(s)(a_x) = \mathsf{P}(s)(b_x) = b_x$, i.e., where $s \in x'$ and $x's \subset x'$. Since $sx' = x's$, we obtain $sx' \subset x'$. If $t \in x'$ and $x't \not\subset x$, then $st \in x$ and sx' cannot be a subset of x' — a contradiction. Therefore, $x't \subset x'$ and $x'x' \subset x'$, i.e., $a_x = b_x$.

As in the proof of Theorem 3, one can verify that $\mathsf{P}(s) \cup \mathsf{P}(t) = \mathsf{P}(s \vee t)$ and $\mathsf{P}(t) \circ \mathsf{P}(s) = \mathsf{P}(st)$, $\mathsf{P}(s) = \mathsf{P}(t) \leftrightarrow s = t$ for all $s, t \in S$. It follows that P is an isomorphism of $(S; \cdot, \vee)$ onto a $\cup$-semigroup of one-to-one functions. The proof of Theorem 4 is complete.

While the problem of determining the structure of ordered semigroups and $\cup$-semigroups of functions remains open, the solution to an analogous problem in case of one-to-one functions follows.

Suppose $(S; \cdot, \leqslant)$ is an ordered semigroup satisfying the conditions of Theorem 2, $I = \{i\colon i^2 = i\}$ is the set of all idempotents of $(S; \cdot)$ and, for every $i \in I$, $S_i = \{s\colon s^2 = i\}$. By Corollary 1 to Theorem 2 from [11], I forms a semilattice (i.e., an idempotent and commutative semigroup) under induced multiplication, and $(S; \cdot)$ is an *inflation* of the semilattice I, i.e., if $s_i \in S_i$ and $s_j \in S_j$, then $s_i s_j = ij$. Let $\dashv$ denote the canonical order relation of the semilattice I. Since I is isomorphic with a semigroup of one-to-one functions in a way such that $\leqslant$ corresponds to the set-theoretic inclusion, $\leqslant$ coincides with $\dashv$ on I [2] (this fact follows readily from the conditions of Theorem 2). By (3), $s^2 \leqslant s$, i.e., i is the smallest element of S_i. If $i \in I$ and $s \leqslant i$, then $s \leqslant ii$ and, by (2), $s \leqslant si$. By (3), $si \leqslant s$, i.e., $s = si$. Since si is idempotent, $s \in I$, i.e., $s \leqslant i \in I$ imply $s \in I$.

Thus we have proved the sufficiency in the following

THEOREM 5. *Let $(I; *)$ be a semilattice with the canonical order $\dashv$, i.e., $i \dashv j \leftrightarrow i * j = i$; let A be a set (possibly empty) and $f\colon A \to I$ a mapping of A into I; let the sets I and A be disjoint. On the set $S = I \cup A$ define a multiplication and an order relation $\leqslant$ in the following way: on I, the multiplication coincides with $*$ and the order $\leqslant$ coincides with $\dashv$; if $s, t \in A$, then $st = f(s) * f(t)$, $is = si = i * f(s)$; $i \leqslant s \leftrightarrow i \dashv s^2$ for every $i \in I$; on A, the order $\leqslant$ is arbitrary. Then $(S; \cdot, \leqslant)$ is an ordered semigroup satisfying the conditions of Theorem 2 and every ordered semigroup satisfying the conditions of Theorem 2 can be constructed in the above-mentioned way.*

Proof. The conditions of Theorem 2 can be verified by a straightforward computation. The converse part of the theorem has been proved above. Theorem 5 is proved.

THEOREM 6. *Let $(I; \wedge, \vee)$ be a distributive lattice and, for every $i \in I$, let $(S_i; \cdot, \vee)$ be an algebra such that $i \in S_i$, $st = i$ for all $s, t \in S_i$, $(S_i; \vee)$ is a semilattice with the smallest element i, and if $i \neq j$, then $S_i \cap S_j = \emptyset$. For any $i, j \in I$ such that $i \leqslant j$ in $(I; \wedge, \vee)$, let f_{ij} be an isomorphism of $(S_i; \cdot, \vee)$ into $(S_j; \cdot, \vee)$, and the isomorphisms form a direct system, i.e.,*

f_{ii} is the identical automorphism and $f_{jk}\circ f_{ij} = f_{ik}$ for all $i, j, k \in I$. On the set $S = \bigcup_{i\in I} S_i$ define two binary operations: if $s_i \in S_i$ and $s_j \in S_j$, then let

$$s_i s_j = i \wedge j \quad \text{and} \quad s_i \vee s_j = f_{i,i\vee j}(s_i) \vee f_{j,i\vee j}(s_j),$$

where $\vee$ in the right-hand side denotes the semilattice operation of $(S_{i\vee j}; \cdot, \vee)$ and $i \vee j$ is the meet of i and j in $(I; \wedge, \vee)$. Then $(S; \cdot, \vee)$ is a semilattice-ordered semigroup satisfying the conditions of Theorem 4 and every semilattice-ordered semigroup satisfying the conditions of Theorem 4 can be constructed in the above-mentioned way.

Proof. The conditions of Theorem 4 can be verified by a routine computation. Conversely, let $(S; \cdot, \vee)$ satisfy the conditions of Theorem 4. By Corollary 1 to Theorem 2 of [11], the set of all idempotents I is a semilattice under the induced multiplication. Denoting by $\wedge$ the induced multiplication on I, we see that $(I; \wedge, \vee)$ is a distributive lattice (here $\vee$ denotes the operation induced on I by the semilattice operation on S). Let $S_i = \{s : s^2 = i\}$. Then the algebras $(S_i; \cdot, \vee)$ satisfy the conditions of Theorem 6. If $i \leqslant j$, then $f_{ij}(s_i) = s_i \vee j$ for all $s_i \in S_i$, and for all $i, j \in I$. Clearly, f_{ij} form a direct system of homomorphisms. To show that f_{ij} is an isomorphism, suppose $f_{ij}(s_i) = f_{ij}(t_i)$ for some $s_i, t_i \in S_i$. Then $s_i \vee j = t_i \vee j$ in $(S; \cdot, \vee)$. Therefore,

$$s_i \leqslant t_i \vee j = t_i \vee jj,$$

whence, by (5),

$$s_i \leqslant t_i \vee s_i j = t_i \vee ij \leqslant t_i,$$

since $ij \leqslant i \leqslant t_i$. Analogously, $t_i \leqslant s_i$, i.e., $s_i = t_i$. If $s_i \in S_i$ and $s_j \in S_j$, then

$$s_i \vee s_j = s_i \vee i \vee j \vee s_j = (s_i \vee i \vee j) \vee (s_j \vee i \vee j) = f_{i,i\vee j}(s_i) \vee f_{j,i\vee j}(s_j).$$

The proof of Theorem 6 is complete.

For the sake of completeness we give here related results proved elsewhere.

THEOREM 7 (see [1]). *A semilattice-ordered semigroup $(S; \cdot, \wedge)$ is isomorphic with a function $\cap$-semigroup if and only if the multiplication $\cdot$ is left distributive relatively to the semilattice meet $\wedge$ and*

$$(6) \qquad (s \wedge t \wedge u)x \wedge (t \wedge u)y = (s \wedge u)x \wedge (t \wedge u)y,$$

where x and y can be either elements of S or empty symbols.

THEOREM 8 (see [1]). *A semilattice-ordered semigroup $(S; \cdot, \wedge)$ is isomorphic with a $\cap$-semigroup of one-to-one functions if and only if the multiplication $\cdot$ is distributive (at both sides) relatively to the semilattice meet $\wedge$ and*

$$(7) \qquad xv \wedge uv \wedge uy \wedge xy = xv \wedge uv \wedge uy,$$

where the variables can be either elements of S or empty symbols, provided identity (7) *makes sense* (*i.e.*, xv, uv, uy, $xy \in S$).

The proofs of these theorems are based on the method of determinative pairs devised in [3] (see also [6]).

If a set F of functions is closed under the composition $\circ$ and both set-theoretic intersection $\cap$ and union $\cup$, then the algebra $(F; \circ, \cap, \cup)$ is called a *function* $\cap\cup$*-semigroup*. Clearly, it is a lattice-ordered semigroup.

THEOREM 9 (see [10]). *A lattice-ordered semigroup* $(S; \cdot, \wedge, \vee)$ *is isomorphic with a function* $\cap\cup$*-semigroup if and only if the multiplication* $\cdot$ *is distributive relatively to both* $\wedge$ *and* $\vee$, *the lattice* $(S; \wedge, \vee)$ *is distributive and the following identity holds*:

$$x \wedge yz = xz \wedge y. \tag{8}$$

THEOREM 10 (see [10]). *A lattice-ordered semigroup* $(S; \cdot, \wedge, \vee)$ *is isomorphic with a* $\cap\cup$*-semigroup of one-to-one functions if and only if it satisfies the conditions of Theorem* 9 *and the semigroup* $(S; \cdot)$ *is commutative*.

THEOREM 11. *Let* $(I; \wedge, \vee)$ *be a distributive lattice and, for every* $i \in I$, *let* $(S_i; \cdot, \wedge, \vee)$ *be a distributive lattice with the smallest element* i *endowed with a multiplication* $\cdot$ *such that* $st = i$ *for all* $s, t \in S_i$, *the subsets* $(S_i)_{i \in I}$ *being disjoint. For every* $i, j \in I$ *such that* $i \leqslant j$, *let* f_{ij} *be an isomorphism of* $(S_i; \cdot, \wedge, \vee)$ *onto an ideal of* $(S_j; \cdot, \wedge, \vee)$ (*i.e., onto an ideal of the lattice* $(S_j; \wedge, \vee)$ *which contains the element* j), *let the isomorphisms form a direct system and, for every* $i, j \in I$,

$$f_{i,i\vee j}(S_i) \cap f_{j,i\vee j}(S_j) = f_{i\wedge j, i\vee j}(S_{i\wedge j}).$$

On the set $S = \bigcup_{i \in I} S_i$ *define three binary operations: if* $s_i \in S_i$ *and* $s_j \in S_j$, *then*

$$s_i s_j = i \wedge j, \qquad s_i \vee s_j = f_{i,i\vee j}(s_i) \vee f_{j,i\vee j}(s_j),$$

and

$$s_i \wedge s_j = f^{-1}_{i\wedge j, i\vee j}\big(f_{i,i\vee j}(s_i) \wedge f_{j,i\vee j}(s_j)\big).$$

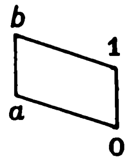
Then $(S; \cdot, \wedge, \vee)$ *satisfies the conditions of Theorem* 10 *and, conversely, every algebra satisfying the conditions of Theorem* 10 *can be constructed in the above-mentioned way*.

Proof. A routine verification shows that the algebra constructed in Theorem 11 is a lattice-ordered semigroup satisfying the conditions of Theorem 10. Conversely, if $(S; \cdot, \wedge, \vee)$ satisfies the conditions of Theorem 10, then $(S; \cdot, \wedge, \vee)$ can be constructed as in Theorem 11, provided $(I; \wedge, \vee)$ is the lattice of all idempotents of the semigroup $(S; \cdot)$ with the induced lattice operations, $S_i = \{s\colon s^2 = i\}$, and $(S_i; \cdot, \wedge, \vee)$ is a subalgebra of $(S; \cdot, \wedge, \vee)$, and $f_{ij}(s_i) = s_i \vee j$ for all $i, j \in I$ such that $i \leqslant j$ and for all $s_i \in S_i$.

Constructions alternative to those in Theorems 6 and 11 can be devised if the algebras S_i are considered as subalgebras of the colimit of the direct system of these algebras.

Remark. Theorems 1-4 and 7-10 state that certain classes of algebras or algebraic systems (those isomorphic with various function semigroups) are axiomatizable by explicitely given systems of axioms. The existence of elementary axiomatics for such classes follows from the main theorem on relation algebras (see [5], [7] and [6]). Axioms given in Theorems 7-10 are identities, i.e., the corresponding classes of algebras are varieties. The external form of the axioms given in Theorems 1-4 shows that the corresponding classes of algebras are quasi-varieties. That they are not varieties one can see from the following

Example. Consider the four-element ordered set given by the diagram



Endow the set with the obvious $\vee$-operation and with the multiplication $0x = a^2 = ab = a1 = 0$ for every x and $b^2 = 1^2 = 1$. Let the multiplication be commutative. We obtain an algebra satisfying the conditions of Theorem 4. Consider the equivalence relation ε on the set: the classes modulo ε are $\{0\}$, $\{a\}$ and $\{1, b\}$. It is easy to verify that ε is a congruence. However, the quotient algebra modulo ε does not satisfy the conditions of Theorems 1-4; therefore, the classes of algebraic systems described in Theorems 1-4 cannot be varieties, since they are not closed under homomorphisms.

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