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CHARACTERIZING THE ARC BY COMPOSITION OF FUNCTIONS

BY

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Let X be a compact metric space and let $C(X)$ be the space of continuous functions of X onto X with metric induced by the sup norm. For f and g belonging to $C(X)$, we set $f < g$ if there is a function a in $C(X)$ such that $fa = g$; this defines a quasi-ordering (a relation that is reflexive and transitive) on $C(X)$. A subset D of $C(X)$ is said to be ***directed*** if, for each pair of functions f and g belonging to D, there is a function h in D such that $f < h$ and $g < h$; which means there exist functions a and β in $C(X)$ such that $fa = g\beta$.

It is known that a Peano continuum X is an arc if and only if $C(X)$ contains a dense directed subset. The "only if" statement in this theorem is proved by Mioduszewski in [6]. That this condition actually characterizes the arc among Peano continua is proved by White in [8]. Here we extend this theorem to a larger class of arcwise connected continua.

Notation. We denote the cone with vertex v over a space T by $T*v$. Recall that the cone over T is obtained from the product space $T\times[0,1]$ by identifying $T\times\{1\}$ to a point v, called the *vertex* of the cone.

Definitions. A *continuum* is a non-degenerate compact connected metric space. A *Peano continuum* is a locally connected continuum. A metric space X is said to be a *quasi-Peano continuum* if there exists a compact totally disconnected metric space T such that $T*v$ is a continuous image of X and X is a continuous image of $T*v$.

It follows from Urysohn's lemma ([1], p. 57) and the Hahn-Mazurkiewicz theorem ([1], p. 129) that every Peano continuum is a quasi-Peano continuum.

In this paper we prove that a quasi-Peano continuum X is an arc if and only if $C(X)$ contains a dense directed set.

Note that since the cone over any non-empty compact totally disconnected metric space is a continuous image of the cone over the Cantor set, every quasi-Peano continuum is a continuous image of the Cantor cone. The following example indicates that not all metric continuous images of the Cantor cone are quasi-Peano continua.

Example. In the complex plane, define F to be the closure of

$$\{re^{i\theta}\colon 0 \leqslant r \leqslant 1 \text{ and } \theta = 2^{-n}\pi, \text{ where } n \text{ is a positive integer}\}.$$

Note that F is a cone over a convergent sequence. Let $\{F_n\}_{n=1}^{\infty}$ be a collection of disjoint cones in the complex plane, each homeomorphic to F such that, for each n, the diameter of F_n is less than 2^{-n} and F_n meets F only at the point $\exp[i\cdot 2^{-n}\pi]$, which is the vertex of F_n (see Fig. 1). Let X be the plane continuum $F \cup \bigcup_{n=1}^{\infty} F_n$.

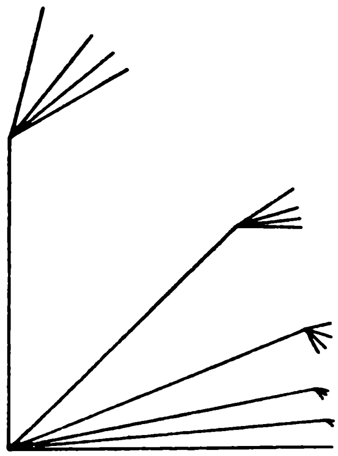

Fig. 1

Note that the Cantor cone can be mapped continuously onto X. In fact, X is a continuous image of the cone over the closure of the planar set of points having rectangular coordinates of the form (m^{-1}, n^{-1}), where m and n are integers and $0 < m \leqslant n$.

If T is a compact totally disconnected metric space that has infinitely many limit points, then X cannot be mapped continuously onto the cone over T. Furthermore, if T is a compact totally disconnected metric space that has only finitely many limit points, then X is not a continuous image of the cone over T. Hence X is not a quasi-Peano continuum.

It is known that the hyperspace of closed subsets of each continuum is a quasi-Peano continuum ([3], Theorem 2.7, and [5]).

Definitions. For topological spaces X and Y, let $C(X, Y)$ denote the collection of continuous functions that take X onto Y, and let ΔY denote the diagonal of $Y \times Y$. Then, for functions f and g belonging to $C(X, Y)$, define the continuous function $f \times g$ of $X \times X$ onto $Y \times Y$ by

$$f \times g(x_1, x_2) = \big(f(x_1), g(x_2)\big);$$

and define the *double graph* of f and g to be the subset $[f, g] = (f \times g)^{-1}(\Delta Y)$ of $X \times X$.

Double graphs are introduced in [2] and [7], and studied in [6] and [8]. As in [8], we use the following three properties of double graphs.

PROPERTY 1. *Let f, g, α and β be elements of $C(X)$; then $f\alpha = g\beta$ if and only if $\alpha \times \beta(\Delta X)$ is a subset of $[f, g]$.*

Indeed, this property follows immediately from the definition.

PROPERTY 2. *If f and g belong to $C(X)$ and U is an open neighborhood of $[f, g]$ in $X \times X$, then there exist neighborhoods V_f of f and V_g of g in $C(X)$ such that $[f', g']$ is a subset of U for every f' in V_f and g' in V_g.*

For the proof see [8], p. 190.

Definition. Let π_1 and π_2 be the projections of $X \times X$ onto its first and second factors, respectively. A subset F of $X \times X$ is said to be *full* if $\pi_1(F) = \pi_2(F) = X$.

PROPERTY 3. *If f, g, α and β belong to $C(X)$ for X connected and $f\alpha = g\beta$, then some component of $[f, g]$ is full.*

Indeed, it follows from Property 1, the continuity of $\alpha \times \beta$, and the fact that α and β map X onto itself, that $\alpha \times \beta(\Delta X)$ is a full connected subset of $[f, g]$.

Notation. Let p and q be distinct points of the plane E^2. We denote the closed straight-line segment in E^2 from p to q by $\langle p, q\rangle$. We assume that the arcs from the vertex in each cone in E^2 (used in the proof of Theorem 1) are straight.

THEOREM 1. *If X is a quasi-Peano continuum and $C(X)$ has a dense directed subset, then X is an arc.*

Proof. Since X is a quasi-Peano continuum, there exist a compact totally disconnected metric space T and continuous functions ψ and μ such that ψ maps X onto $T * v$ and μ maps $T * v$ onto X. There exists a compact subset T_1 of T such that

(1) μ takes the subcone $T_1 * v$ of $T * v$ onto X, and

(2) for each proper compact subset Z of T_1, the subcone $Z * v$ of $T * v$ is not mapped by μ onto X.

Let σ_1 be a continuous function of $T * v$ onto $T_1 * v$.

Assume that T_1 is an infinite set. By considering three cases, we prove that this assumption contradicts the hypothesis of the theorem. This suffices to prove the theorem, since T_1 being finite implies that X is a Peano continuum ([9], Theorem 1.51, p. 26) and, therefore, an arc ([8], Theorem 1).

Case 1. Suppose that T_1 is a perfect set. For $i = 1, 2, 3, 4$, define C_i to be the Cantor ternary set on the interval $\langle(2i-1, 0), (2i, 0)\rangle$

in E^2. Define C to be the plane continuum

$$\langle(2,1),(8,1)\rangle \cup \bigcup_{i=1}^{4} C_i * (2i,1).$$

Since T_1 is a compact totally disconnected perfect metric space, there is a continuous function σ_2 of $T_1 * v$ onto C.

For $i = 1, 2$, let D_i be the Cantor ternary set on the interval $\langle(0, 2i-1), (1, 2i-1)\rangle$; and, for $i = 3, 4$, let D_i be the Cantor ternary set on $\langle(-1, 2i-3), (0, 2i-3)\rangle$. Define D to be the plane continuum

$$(D_1 \cup D_2) * (0,2) \cup (D_3 \cup D_4) * (0,4).$$

Define σ_3 to be a continuous function that takes C onto D.

Note that $\varphi = \sigma_3\sigma_2\sigma_1\psi$ is a continuous function of X onto D.

There exist mutually exclusive perfect subsets Z_1, Z_2 and Z_3 of T_1 such that

(1) $\bigcup_{i=1}^{3} Z_i = T_1$, and

(2) for each $i = 1, 2, 3$, there is a point p_i of the subcone $Z_i * v$ of $T_1 * v$ with the property that $\mu(p_i)$ does not belong to $\mu\big((T_1 \setminus Z_i) * v\big)$.

For each $i = 1, 2$, let h_i be a homeomorphism of $D_i * (0, 2)$ onto $Z_i * v$ that takes some point a_i of $\langle(0, i), (0, i+1)\rangle$ to p_i; and, for each $i = 3, 4$, define h_i to be a homeomorphism of $D_i * (0, 4)$ onto $Z_{i-1} * v$ that takes a point a_i of $\langle(0, i), (0, i+1)\rangle$ to p_{i-1}.

For each $i = 1, 2$, let k_i be a homeomorphism of $D_i * (0, 2)$ onto $Z_{2i-1} * v$ that takes a point b_i of $\langle(0, i), (0, i+1)\rangle$ to p_{2i-1}; and, for each $i = 3, 4$, let k_i be a homeomorphism of $D_i * (0, 4)$ onto $Z_{6-i} * v$ that takes a point b_i of $\langle(0, i), (0, i+1)\rangle$ to p_{6-i}.

Using these homeomorphisms, we now define continuous functions γ and ϱ of D onto $T_1 * v$. For each point z of D, select integers i and j such that z belongs to the domain of h_i and to the domain of k_j, and then define γ and ϱ by $\gamma(z) = h_i(z)$ and $\varrho(z) = k_j(z)$.

It follows that $f^* = \mu\gamma$ and $g^* = \mu\varrho$ are continuous functions of D onto X.

No component of $[f^*, g^*]$ is full. To see this assume there exists a component F of $[f^*, g^*]$ in $D \times D$ that is full. Note that the set $\{(0,3)\} \times D$ separates $D \times D$. Let H denote the closure of the component of the set $D \times D \setminus \big(\{(0,3)\} \times D\big)$ containing $\{a_1\} \times D$. Let J be a component of $F \cap H$ with the property that a_1 belongs to $\pi_1(J)$. Since J meets $\{(0,3)\} \times D$ ([4], Theorem 1, p. 172), the continuum $\pi_1(J)$ in D contains a_2. Let c_1 and c_2 be points of D such that (a_1, c_1) and (a_2, c_2) belong to J. Since J is a subset of $[f^*, g^*]$, it follows that

$$\mu\gamma(a_1) = \mu\varrho(c_1) = \mu(p_1) \quad \text{and} \quad \mu\gamma(a_2) = \mu\varrho(c_2) = \mu(p_2).$$

Since, for $i = 1, 2$, the point $\mu(p_i)$ is not in $\mu\big((T_1 \setminus Z_i) * v\big)$, the points c_1 and c_2 must belong to $D_1 * (0, 2)$ and $D_4 * (0, 4)$, respectively. Hence the continuum $\pi_2(J)$ intersects both $D_1 * (0, 2)$ and $D_4 * (0, 4)$ and, therefore, contains the point b_2. But this implies the existence of a point z of $\pi_1(H)$ such that (z, b_2) belongs to J, which is impossible, since $g^*(b_2)$ is not a point of $f^*\big(\pi_1(H)\big)$.

Following White ([8], the proof of Theorem 1, p. 191) we now write $f = f^*\varphi$ and $g = g^*\varphi$. Observe that since $\varphi \times \varphi([f, g])$ is a subset of $[f^*, g^*]$ and $\varphi\pi_i = \pi_i(\varphi \times \varphi)$, no component of $[f, g]$ in $X \times X$ is full. It follows from a short argument of White ([8], the proof of Theorem 1, p. 191) involving Property 2 that there exist neighborhoods V_f of f and V_g of g in $C(X)$ such that no component of $[f', g']$ is full if f' belongs to V_f and g' belongs to V_g. According to Property 3, this contradicts the hypothesis of the theorem.

Case 2. Suppose that T_1 has exactly one isolated point y. By adjoining the closed segment $\langle(\frac{1}{2}, 1), (0, 2)\rangle$ to the set D (defined in Case 1), extending γ and ϱ to continuous functions each of which takes $\langle(\frac{1}{2}, 1), (0, 2)\rangle$ onto the arc in $T_1 * v$ from y to v, and then applying the preceding argument, we get a contradiction.

Case 3. Suppose that T_1 has more than one isolated point. Let y_1 and y_2 be distinct isolated points of T_1. Define E_1 to be a subset of the closed interval $\langle(0, 1), (1, 1)\rangle$ that contains the point $(0, 1)$ and is homeomorphic to $T_1 \setminus \{y_1, y_2\}$. Let E denote the planar set

$$\langle(0, 2), (0, 5)\rangle \cup E_1 * (0, 2).$$

Applying the argument of Case 1, with E in place of D and the arcs in $T_1 * v$ from v to y_1 and y_2 in place of $Z_2 * v$ and $Z_3 * v$, we again get a contradiction.

Hence T_1 is finite and the cone $T_1 * v$ is locally connected. Therefore X, being a continuous image of $T_1 * v$, is a Peano continuum ([9], Theorem 1.51, p. 26). It follows from White's theorem ([8], Theorem 1) that X is an arc.

THEOREM 2. *Suppose that X is a quasi-Peano continuum. Then X is an arc if and only if $C(X)$ contains a dense directed subset.*

This theorem follows directly from Theorem 1 and [6], Theorem 2.

REFERENCES

[1] J. Hocking and G. Young, *Topology*, Massachusetts 1961.

[2] T. Homma, *A theorem on continuous functions*, Kodai Mathematical Seminar Reports 1 (1952), p. 13-16.

[3] J. L. Kelley, *Hyperspaces of a continuum*, Transactions of the American Mathematical Society 52 (1942), p. 22-36.
[4] K. Kuratowski, *Topology*, Vol. 2, New York 1968.
[5] S. Mazurkiewicz, *Sur le type c de l'hyperspace d'un continu*, Fundamenta Mathematicae 20 (1933), p. 52-53.
[6] J. Mioduszewski, *On a quasi-ordering in the class of continuous mappings of a closed interval into itself*, Colloquium Mathematicum 9 (1962), p. 233-240.
[7] R. Sikorski and K. Zarankiewicz, *On uniformization of functions (I)*, Fundamenta Mathematicae 41 (1954), p. 339-344.
[8] W. White, *Characterizing the interval and the circle by composition of functions*, Colloquium Mathematicum 24 (1972), p. 189-193.
[9] G. T. Whyburn, *Analytic topology*, American Mathematical Society Colloquium Publications 28 (1963).

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