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VARIETIES AND FINITE CLOSURE CONDITIONS

BY

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We deal throughout with varieties of a fixed finite similarity type τ. In addition to the usual notations $H(K)$, $S(K)$, $P(K)$ and $S_P(K)$ we use $P^F(K)$ ($S_P^F(K)$) to denote the class of finite products (finite subdirect products) of members of K. For all other notation consult [5]. We call a set K of finite algebras *finitely closed* if $HSP^F(K) = K$. Andreas Blass asked if the finitely closed sets could be characterized. Let K_ω denote the finite members of a class K of algebras. We show that K is finitely closed if and only if

$$K = \bigcup_{i<\omega} (V^i)_\omega$$

for some sequence of varieties V^i. This leads us to a more general discussion of the relationship between a variety, its finite members, and finitely closed classes. This discussion is facilitated by denoting $HSP(K)$ by K^*. Thus for any class of algebras K we have two operators * and sub-ω which can easily be seen to satisfy

(i) $(K^*)^* = K^*$ and $(K_\omega)_\omega = K_\omega$,

(ii) $K_\omega \subseteq K$, $K \subseteq K^*$,

(iii) $K_1 \subseteq K_2$ implies $K_1^* \subseteq K_2^*$ and $(K_1)_\omega \subseteq (K_2)_\omega$, and (somewhat less easily),

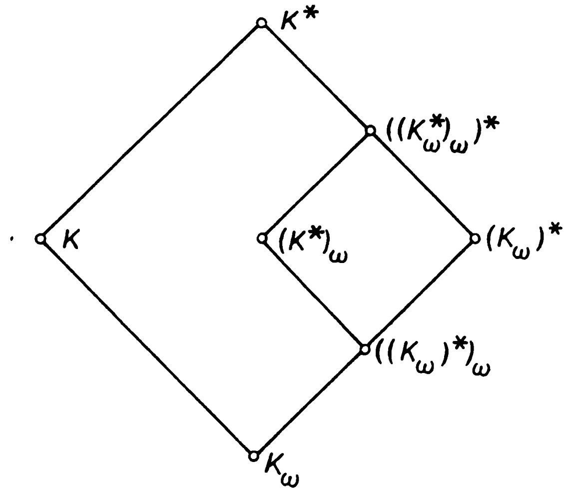
(iv) $(K^*)_\omega = \big(((K^*)_\omega)^*\big)_\omega$ and $(K_\omega)^* = \big(((K_\omega)^*)_\omega\big)^*$.

Subject to these conditions, a given class K can generate at most 7 classes under the operations * and sub-ω. They are partially ordered by inclusion as shown in the diagram on the next page.

This diagram is realized if $K = \{F_V(\omega), \mathscr{A}\}$, where V is a variety of lattices which is not generated by its finite members, $F_V(n)$ a free algebra on n free generators in V, and $\mathscr{A}$ the 2-element lattice. Examples of varieties V have been found in [1] and [8].

If K is a variety, the diagram becomes $K_\omega \subseteq (K_\omega)^* \subseteq K$, while if K contains only finite algebras, the diagram becomes $K \subseteq (K^*)_\omega \subseteq K^*$. Some of our results can be interpreted in terms of further simplifications of the diagram.

We can now formulate a "dual" to the well-known problem (see [1], [3], [4] and [8]) of when a variety is generated by its finite members (in our notation when does $(V_\omega)^* = V$).



QUESTION 1. Give necessary and sufficient conditions on a set K of finite algebras for $(K^*)_\omega$ to equal K. (**P 965**)

We give a sufficient but not necessary condition in Theorem 2.

We require the following variant of Birkhoff's Theorem. It can be proved by examining the proof in [5], Theorem 1, p. 167. In the following we denote by K_n, $n < \omega$, those members of a class K of algebras which are generated by n or fewer elements.

LEMMA 1. *Let K be a class of algebras. Then $F_{K^*}(n)$ is in $S_P(S(K_n))$. Moreover, if $S(K_n)$ is finite, then $F_{K^*}(n) \in S_P^F(S(K_n))$.*

A class K of algebras is *locally finite* if every finitely generated member of K is finite. The class K is *uniformly locally finite* if K has finite algebras with more than one element and there is a function $f\colon \omega \to \omega$ such that every n generated algebra in K has not greater than $f(n)$ elements. A locally finite variety is easily seen to be uniformly locally finite by taking $f(n) = |F_K(n)|$.

We can now characterize finitely closed sets of algebras.

THEOREM 1. *Let K be a set of finite algebras. The following are equivalent:*

(i) $K = HS_P^F(S(K))$;

(ii) $K = HSP^F(K)$;

(iii) *there exists a sequence of varieties $\langle M_n\rangle_{n<\omega}$ such that $M_n \subseteq M_{n+1}$ and $K = \bigcup (M_n)_\omega$.*

Proof. It is obvious that (i)→(ii) and (iii)→(i); we show that (ii)→(iii). Suppose $K = HSP^F(K)$ and let K'_n be $\{\mathscr{A} \in K \mid |A| \leqslant n\}$ and $M_n = HSP(K'_n)$.

It follows from Lemma 1 that

$$(M_n)_\omega \subseteq HS_P^F(S(K_n')) = HS_P^F(K_n') \subseteq HSP^F(K) \subseteq K,$$

since each $S(K_n')$ is finite. Thus $\bigcup (M_n)_\omega \subseteq K$ while, clearly, $K \subseteq \bigcup (M_n)_\omega$ and $M_n \subseteq M_{n+1}$.

We now return to Question 1. The following is similar to 14.3 of [6]:

THEOREM 2. *Let K be a set of finite algebras containing at least one algebra with more than one element. The following are equivalent:*

(i) *K is uniformly locally finite and finitely closed;*

(ii) *K^* is locally finite and $K = (K^*)_\omega$;*

(iii) *$K = V_\omega$ for some locally finite variety V.*

Proof. (ii) implies (iii) and (iii) implies (i) are clear. We prove (i) implies (ii). Note that, for each $n < \omega$, K_n is finite, since K is uniformly locally finite. Hence, by Lemma 1, $F_{K^*}(n) \in S_P^F(S(K_n))$ and $S_P^F(S(K_n)) \subseteq K$, since K is finitely closed. Thus every finitely generated member of K^* is in $H(K) = K$ and is finite, proving the theorem.

Thus we have obtained a sufficient condition in answer to Question 1. If K is finitely closed and uniformly locally finite, then $K = (K^*)_\omega$. Clearly, it is necessary that K be finitely closed and, equally clearly (e.g., lattices), it is not necessary that K be uniformly locally finite.

The general question of when a variety is generated by its finite members can be phrased as

(i) give conditions on V such that $(V_\omega)^* = V$ or

(ii) give conditions on V_ω such that $(V_\omega)^* = V$ or

(iii) give conditions on V and V_ω such that $(V_\omega)^* = V$.

Clearly, if V is locally finite, then $(V_\omega)^* = V$. Thus if V is generated by a finite algebra, then V is locally finite, and hence $(V_\omega)^* = V$. Moreover, if V is any variety such that V_ω is uniformly locally finite and V is residually finite, then V is locally finite and $(V_\omega)^* = V$. (The uniform local finiteness of V_ω is necessary, see [2].)

A variety V has the *finite embedding property* if every finite partial algebra $\mathfrak{A}$ which can be embedded in a member of V can be embedded in a finite member of V. Evans proved ([4], Theorem 4; his result is actually stronger than this) that any variety V with the finite embedding property satisfies $(V_\omega)^* = V$. Clearly, any locally finite variety has the finite embedding property but not conversely (e.g., lattices). We can obtain a partial converse.

THEOREM 3. *For any variety V, the following are equivalent:*

(i) *V is locally finite;*

(ii) *V_ω is uniformly locally finite and V has the finite embedding property;*

(iii) *there is some finitely closed and uniformly locally finite set* K *such that* $V = K^*$.

Proof. Obviously, (i) → (ii) and, by Theorem 2, (i) → (iii). To see (ii) → (i) note that assuming (ii), by the Evans result, $(V_\omega)^* = V$ and, by Theorem 2, $(V_\omega)^*$ is locally finite.

The question arises: Can Theorem 3 be strengthened by replacing condition (ii) by

(ii′) V_ω is uniformly locally finite?

We now show the answer is no.

Let V be the variety of lattice ordered groups. The only finite member of V is the 1-element algebra, so V is not locally finite and V is not generated by its finite members. Only our proviso that a uniformly locally finite class must have non-trivial finite members prevents V_ω from being uniformly locally finite. In the mentioned earlier non-trivial examples of varieties which are not generated by their finite members, V_ω is not uniformly locally finite. This raises the following question:

Does there exist a variety V which is not locally finite, but with V_ω uniformly locally finite?

Walter Taylor has provided the following example which we include here with his permission. The product $V \otimes W$ of a pair of varieties is defined in [7]. For our purposes it is enough to remark that if $\mathscr{A} \in V \otimes W$, then there exist $\mathscr{B} \in V$ and $\mathscr{C} \in W$ such that $A = B \times C$ ([7], 1.15). Now let V be any variety which has no non-trivial finite members, e.g., lattice ordered groups, and W any locally finite variety, e.g., distributive lattices. Then any finite member $\mathfrak{A}$ of $V \otimes W$ must have $A = \{1\} \times C$ for some finite member $\mathscr{C}$ of W. Clearly, $(V \otimes W)_\omega$ is uniformly locally finite but $V \otimes W$ is not locally finite.

Careful examination of the definition of $V \otimes W$ shows that if V and W are of finite type, then $V \otimes W$ is polynomially definitionally equivalent to a variety of finite type.

In the remainder of the paper, we explore the properties of varieties V such that V_ω is uniformly locally finite but V is not locally finite.

THEOREM 4. *Let* V *be a variety such that* V_ω *is uniformly locally finite. Exactly one of the following conditions holds:*

(i) V *is locally finite*;

(ii) *for some* n, *there exists an infinite set* P *of* n-*ary polynomials such that, for each* $p, q \in P$,

$$V_\omega \models p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n)$$

but for every pair $p, q \in P$ *there exist an infinite* n-*generated subdirectly irreducible algebra* $\mathscr{A}$ *in* V *and elements* $a_1, \ldots, a_n \in \mathscr{A}$ *such that*

$$\mathscr{A} \models p(a_1, \ldots, a_n) \neq q(a_1, \ldots, a_n).$$

Proof. Clearly, (i) and (ii) are mutually exclusive. Suppose V is not locally finite. Then, for some n, $F_V(n)$ is infinite. Since V_ω is uniformly locally finite, there is a finite algebra $\mathscr{B}$ in V which is n-generated and has the largest cardinality of any finite n-generated algebra in V. Then, for some congruence θ on $F_V(n)$, $\mathscr{B} \cong F_V(n)/\theta$. Since $\mathscr{B}$ is finite, some congruence class $[b]\theta$ of θ is infinite. Let P be the set of n-ary polynomials such that $p(x_1, \ldots, x_n) \in [b]\theta$, where $x_1, \ldots, x_n$ generate $F_V(n)$. Now let $\mathscr{C} \in V_\omega$ and suppose, for some p, $q \in P$,

$$\mathscr{C} \models \exists y_1, \ldots, \exists y_n \; p(y_1, \ldots, y_n) \neq q(y_1, \ldots, y_n).$$

Let φ be the congruence determined by the map from $F_K(n)$ into the subalgebra of $\mathscr{C}$ generated by $\{y_1, \ldots, y_n\}$ which takes x_i to y_i for $i \leqslant n$. Since

$$p(x_1, \ldots, x_n) \equiv q(x_1, \ldots, x_n) \; (\theta)$$

but $p(x_1, \ldots, x_n) \not\equiv q(x_1, \ldots, x_n) \; (\varphi)$, we have $\varphi \wedge \theta < \theta$. Hence $F_V(n)/\varphi \wedge \theta = \mathscr{C}'$ has more elements than $\mathscr{B}$. But since both φ and θ have only finitely many congruence classes, so does $\varphi \wedge \theta$. Thus $\mathscr{C}'$ is finite contradicting the maximality of $\mathscr{B}$. Now, for any p, $q \in P$, if γ is a maximal congruence of $F_V(n)$ which separates $p(x_1, \ldots, x_n)$ and $q(x_1, \ldots, x_n)$, then $F_V(n)/\gamma$ is the required infinite subdirectly irreducible algebra.

The following corollary sharpens this result:

COROLLARY. *Let V be a variety which is not locally finite but such that V_ω is uniformly locally finite. There is an integer n such that one of the following two conditions holds:*

(i) *There are infinitely many infinite n-generated subdirectly irreducible algebras in V.*

(ii) *There is an infinite set Q of n-ary polynomials and an infinite subdirectly irreducible algebra $\mathscr{A} \in K$ generated by $a_1, \ldots, a_n$ such that*

$$V_\omega \models p(x_1, \ldots, x_n) = q(x_1, \ldots, x_n) \quad \textit{for every } p, q \in Q$$

and

$$\mathscr{A} \models p(a_1, \ldots, a_n) \neq q(a_1, \ldots, a_n) \quad \textit{for every } p, q \in Q.$$

Proof. Choose $F_K(n)$ and the set P of n-ary polynomials according to (ii) of Theorem 4. Suppose $\mathscr{A}_1, \ldots, \mathscr{A}_n$ enumerate the n-generated infinite subdirectly irreducible algebras in K. Decompose $P^{(2)}$ (the 2-element subsets of P) into $\bigcup_{i=1}^{m} P_i$, where $\{p, q\}$ is in P_i if $\mathscr{A}_i \models p(a_1^i, \ldots, a_n^i) \neq q(a_1^i, \ldots, a_n^i)$ with $a_1^i, \ldots, a_n^i$ generating $\mathscr{A}_i$. By Ramsey's theorem, there exist an infinite set Q and an $\mathscr{A}_i$ such that

$$\mathscr{A}_i \models p(a_1^i, \ldots, a_n^i) \neq q(a_1^i, \ldots, a_n^i) \quad \text{for any } p, q \in Q$$

which proves the Corollary.

REFERENCES

[1] K. Baker, *Equational classes of modular lattices*, Pacific Journal of Mathematics 28 (1969), p. 9-15.

[2] J. T. Baldwin and J. Berman, *The number of subdirectly irreducible algebras in a variety*, preprint.

[3] W. Blok, $2^{\aleph_0}$ *varieties of Heyting algebras not generated by their finite members*, preprint.

[4] T. Evans, *Some connections between residual finiteness, finite embeddability and the word problem*, Journal of the London Mathematical Society 2 (1969), p. 399-403.

[5] G. Grätzer, *Universal algebra*, Princeton, New Jersey, 1968.

[6] A. I. Mal'cev, *Algebraic systems*, Berlin - New York 1973.

[7] W. Taylor, *Characterizing Mal'cev conditions*, Algebra Universalis (to appear).

[8] R. Wille, *On primitive classes of lattices*, American Mathematical Society Notices 15 (1968), p. 781.

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