

*CARISTI'S FIXED POINT THEOREM
AND METRIC CONVEXITY*

BY

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Let M be a metric space. We shall use here * the standard notation of distance geometry with juxtaposition, ab denoting the distance of points a, b in M and (xyz) denoting the relation “ y is between x and z ” (cf. Blumenthal [3], Chapter II). Thus (xyz) indicates $x \neq y \neq z$ and $xy + yz = xz$. The space M is said to be *convex* if given any two points x, z in M with $x \neq z$ there exists a point y in M such that (xyz) . Menger's convexity theorem, stated below, is fundamental in the study of the geometry of metric spaces. It asserts that each two points of a complete and convex metric space M are actually joined by a metric segment of M (that is, by an isometric image of a real line interval whose length equals the distance between the points). It is our object in this somewhat expository note to give a new and simple proof of this basic fact, a proof based upon an easy application of a newly discovered fixed point theorem due to James Caristi. We also show that the fixed point property asserted in Caristi's theorem is characteristic of complete metric spaces, and then we append a simple proof of Caristi's theorem.

THEOREM (Menger [14]). *Any two points of a complete and convex metric space M can be joined by a metric segment of M .*

We derive this theorem from the following:

THEOREM 1 (Caristi [9]). *Let M be a complete metric space, $G: M \rightarrow M$ an arbitrary function, and $\varphi: M \rightarrow R^+$ lower semicontinuous (R^+ denotes the non-negative real numbers). If, for each x in M ,*

$$xG(x) \leq \varphi(x) - \varphi(G(x)),$$

then G has a fixed point in M .

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Before proceeding, some comments about Theorem 1 are in order.

The formulation of Theorem 1 along with Caristi's transfinite induction proof [9] evolved from his study of inward mappings (also see [10]). Further applications of Theorem 1, specifically applications to Browder's theory of normal solvability ([7], [8]), are given in Kirk-Caristi [13] and Kirk [12]. We also note that Theorem 1 is actually equivalent to a theorem announced in 1972 by Ekeland [11] (Théorème 1). Ekeland's result (which is not formulated as a fixed point theorem) is an abstraction of a lemma due to Bishop and Phelps [2]. After suggesting a possible connection between Theorem 1 and the Bishop-Phelps approach, Felix Browder devised a proof [8] of Theorem 1 which avoids completely the use of transfinite induction (and even avoids the axiom of choice). The simple proof we append to this note, based upon an application of Zorn's lemma, is similar to Browder's approach and is implicit in a recent paper of Brøndsted [5]. Finally, we remark that Wong [16] has considerably simplified Caristi's original transfinite induction approach.

Proof of Menger's theorem. We prove this theorem by first establishing two lemmas. Our method basically differs from Menger's original approach only in that we apply Caristi's theorem to give a quick proof of Lemma 1. This lemma subsumes the "deep" part of Menger's argument, a step established originally by transfinite induction. (We should remark that a different proof of Menger's theorem due to Aronszajn [1] can be found in Blumenthal [3], p. 41-43, and in Blumenthal-Menger [4], p. 244-246. Another proof is sketched in Menger-Milgram [15]. Our proof should be compared with these approaches.)

Our first lemma does not require convexity of the space.

LEMMA 1. *Let M be a complete metric space with $a, b \in M$, $a \neq b$, and suppose $0 < \lambda < ab$. Let*

$$B(a, b) = \{x \in M : (axb)\},$$

$$S = S(a, b, \lambda) = \{x \in B(a, b) : ax \leq \lambda\} \cup \{a\}.$$

Then there exists a point $x_\lambda \in M$ such that

- (i) $x_\lambda \in S(a, b, \lambda)$,
- (ii) $z \in B(a, b) \wedge (ax_\lambda z) \Rightarrow az > \lambda$.

Proof. Case 1. There exists $\bar{x} \in S$ with $a\bar{x} < \lambda$ such that $(a\bar{x}z) \Rightarrow z \notin S$. In this case take $x_\lambda = \bar{x}$ and note that (ii) follows trivially.

Case 2. For each $x \in S$ with $ax < \lambda$ there exists $y_x \in S$ such that (axy_x) . In this case define $G: S \rightarrow S$ by taking $G(x) = y_x$ if $ax < \lambda$ and $G(x) = x$ otherwise. Define $\varphi: S \rightarrow R^+$ by $\varphi(x) = \lambda - ax$. Then, clearly, φ is continuous and, for $x \in S$,

$$xG(x) = aG(x) - ax = \lambda - ax - (\lambda - aG(x)) = \varphi(x) - \varphi(G(x)).$$

Since S is closed (hence complete), by Theorem 1 $G(\bar{x}) = \bar{x}$ for some point $\bar{x} \in S$. This implies $a\bar{x} = \lambda$ and $\bar{x} = x_\lambda$ satisfies (i) and (ii).

The remainder of our proof, included here for the sake of completeness, is elementary and follows Menger [14]. We use the following transitivity of betweenness relation (see [3], p. 33):

For points $p, q, r, s \in M$,

$$(pqr) \text{ and } (prs) \Leftrightarrow (pqs) \text{ and } (qrs).$$

LEMMA 2. Let M be complete and convex, $a, b \in M$, $a \neq b$, and suppose $0 < \lambda < ab$. Then there exists $\bar{x} \in M$ such that $(a\bar{x}b)$ and $a\bar{x} = \lambda$.

Proof. By Lemma 1 there exists $x_\lambda \in M$ such that:

- (i) $x_\lambda \in S(a, b, \lambda)$,
- (ii) $z \in B(a, b) \wedge (ax_\lambda z) \Rightarrow az > \lambda$.

Let $\lambda' = ab - \lambda$ and again apply Lemma 1 to obtain $y_{\lambda'} \in M$ such that:

- (i)' $y_{\lambda'} \in S(b, x_\lambda, \lambda')$,
- (ii)' $z \in B(b, x_\lambda) \wedge (by_{\lambda'} z) \Rightarrow bz > \lambda'$.

Case 1. $x_\lambda = y_{\lambda'}$. Then, since $ab = ax_\lambda + x_\lambda b \leq \lambda + \lambda' = ab$, it follows that $ax_\lambda = \lambda$.

Case 2. $x_\lambda \neq y_{\lambda'}$. In this case use convexity of M to obtain $w \in M$ such that $(x_\lambda w y_{\lambda'})$. By assumption the relations $(ax_\lambda b)$, $(x_\lambda y_{\lambda'} b)$, and $(x_\lambda w y_{\lambda'})$ hold. It follows immediately from transitivity of betweenness that (awb) , $(ax_\lambda w)$, $(bw x_\lambda)$, and $(by_{\lambda'} w)$ also hold. Now $(awb) \wedge (ax_\lambda w) \Rightarrow aw > \lambda$ by (ii) and $(bw x_\lambda) \wedge (by_{\lambda'} w) \Rightarrow bw > \lambda'$ by (ii)'. Therefore

$$ab = aw + wb > \lambda + \lambda' = ab.$$

This contradiction establishes Lemma 2 via Case 1.

Proof of Menger's theorem completed. Let $a_0, a_1 \in M$, $a_0 \neq a_1$. By Lemma 2 there exists $a_{1/2} \in M$ such that $a_0 a_{1/2} = a_{1/2} a_1 = \frac{1}{2} a_0 a_1$ (i.e., $a_{1/2}$ is a "midpoint" of the pair (a_0, a_1)). Let $d = a_0 a_1$ and define the mapping F by

$$F(0) = a_0, \quad F\left(\frac{d}{2}\right) = a_{1/2}, \quad F(d) = a_1.$$

Again by Lemma 2 there exist points $a_{1/4}, a_{3/4}$ which are respective midpoints of $(a_0, a_{1/2}), (a_{1/2}, a_1)$. Define

$$F\left(\frac{d}{4}\right) = a_{1/4}, \quad F\left(3\frac{d}{4}\right) = a_{3/4},$$

and use transitivity of betweenness to conclude that F is an isometry on

$$\left\{0, \frac{d}{4}, \frac{d}{2}, 3\frac{d}{4}, d\right\}.$$

By induction obtain points $\{a_{p/2^n}\}$, $1 \leq p \leq 2^n - 1$ ($n = 1, 2, \dots$), in M such that the mapping $F: p_{d/2^n} \rightarrow a_{p/2^n}$ is an isometry. Since $\{p_{d/2^n}\}$ is a dense subset of $(0, d]$ with F an isometry defined on this set, and since M is complete, it is possible to extend F in the obvious way to all of $[0, d]$. The resulting set $F([0, d])$ is a metric segment in M joining a_0 and a_1 . This completes the proof.

We now prove a theorem which together with Theorem 1 characterizes completeness.

THEOREM 2. *Let M be a metric space which is not complete. Then there exist a fixed-point free function $G: M \rightarrow M$ and a continuous mapping $\varphi: M \rightarrow R^+$ such that*

$$xG(x) \leq \varphi(x) - \varphi(G(x)), \quad x \in M.$$

Proof. Let $\{x_n\} \subset M$ be a Cauchy sequence which has no limit. Define $\psi: M \rightarrow R^+$ by

$$\psi(z) = \lim_{i \rightarrow \infty} zx_i, \quad z \in M.$$

Given $x \in M$, let n be the smallest positive integer such that

$$(*) \quad 0 < \frac{1}{2}xx_n \leq \psi(x) - \psi(x_n).$$

(Note that $\psi(x_n) \rightarrow 0$ while $\psi(x) > 0$.) With n so determined, define $G(x) = x_n$ and let $\varphi(x) = 2\psi(x)$. Then, from (*),

$$xG(x) \leq \varphi(x) - \varphi(G(x)).$$

Remark 1. We should point out that if $T: M \rightarrow M$ is a *contraction mapping* (i.e., if there exists $k < 1$ such that $TxTy \leq kxy$, $x, y \in M$), then T satisfies the assumptions of Theorem 1 with $\varphi(x) = (1-k)^{-1}xTx$. We emphasize that the mapping G of Theorem 1 is not assumed to be continuous.

Remark 2. In connection with Theorem 2, we point out that there exist non-complete metric spaces M which have the property that every contraction mapping $T: M \rightarrow M$ has a fixed point.

Proof of Theorem 1 (cf. Brøndsted [5]). For $a, b \in M$, define the relation

$$a \preceq b \Leftrightarrow ab \leq \varphi(a) - \varphi(b).$$

It is easy to verify that (M, \preceq) is a partially ordered set. Fix $\bar{x} \in M$ and use Zorn's lemma to obtain a maximal (relative to set inclusion)

totally ordered subset E of M containing \bar{x} . Assume $E = \{x_\alpha\}_{\alpha \in I}$, where I is totally ordered and

$$x_\alpha \leq x_\beta \Leftrightarrow \alpha \leq \beta \quad (\alpha, \beta \in I).$$

Now $\{\varphi(x_\alpha)\}_{\alpha \in I}$ is a decreasing net in R^+ , so there exists $r \geq 0$ such that $\varphi(x_\alpha) \rightarrow r$ as $\alpha \uparrow$. Let $\varepsilon > 0$. There exists an $\alpha_0 \in I$ such that

$$\alpha \geq \alpha_0 \Rightarrow r \leq \varphi(x_\alpha) \leq r + \varepsilon.$$

Therefore, if $\beta \geq \alpha \geq \alpha_0$, then

$$x_\alpha x_\beta \leq \varphi(x_\alpha) - \varphi(x_\beta) \leq \varepsilon$$

and this proves that $\{x_\alpha\}_{\alpha \in I}$ is a Cauchy net in M . By completeness there exists $x \in M$ such that $x_\alpha \rightarrow x$ as $\alpha \uparrow$. Since φ is lower semicontinuous, $\varphi(x) \leq r$. Also, for $\beta \geq \alpha$,

$$x_\alpha x_\beta \leq \varphi(x_\alpha) - \varphi(x_\beta)$$

and, letting $\beta \uparrow$,

$$x_\alpha x \leq \varphi(x_\alpha) - r \leq \varphi(x_\alpha) - \varphi(x)$$

yielding $x_\alpha \leq x$, $\alpha \in I$. Since E is maximal, $x \in E$. But also

$$xG(x) \leq \varphi(x) - \varphi(G(x)),$$

so it follows that

$$x_\alpha \leq x \leq G(x), \quad \alpha \in I,$$

and, by maximality, $G(x) \in E$. Therefore $G(x) \leq x$ and it follows that $G(x) = x$.

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