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ON SUBSPACES OF THE PIXLEY-ROY EXAMPLE

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Notation. Our set-theoretical and topological terminology is standard. If X is a set, then $|X|$ is the cardinality of X. The set of real numbers is denoted by R. Natural numbers are denoted by n, i, m and k. By λ we denote a cardinal number.

Introduction. Suppose we are given a topological space S. Then by $\mathfrak{X}(S)$ we denote the space of all finite non-void subsets of S with the topology generated by sets $\langle x, V\rangle = \{y \in \mathfrak{X}(S): x \subset y \subset V\}$, where $x \in \mathfrak{X}(S)$ and V is open in S. The space $\mathfrak{X}(R)$ was introduced by Pixley and Roy in [6]. Spaces $\mathfrak{X}(S)$, for $S \subset R$, were considered by Przymusiński and Tall in [8] and [7]. The following facts can be found in the above-quoted papers:

1. Sets $\langle x, V\rangle$ form a base for $\mathfrak{X}(S)$.
2. If $S \subset P$, then $\mathfrak{X}(S)$ is a closed subspace of $\mathfrak{X}(P)$.
3. If S is a T_1-space, then $\mathfrak{X}(S)$ is a T_1-space, sets $\langle x, V\rangle$ are closed, and thus $\mathfrak{X}(S)$ is completely regular.
4. If S is a first countable T_1-space, then $\mathfrak{X}(S)$ is a Moore space.
5. The density of $\mathfrak{X}(S)$ equals the cardinality of S.
6. If P is metrizable and has cellularity λ, then $\mathfrak{X}(P)$ has cellularity λ.
7. If Martin's Axiom (MA) is assumed, then each subspace Y of $\mathfrak{X}(R)$, $|Y| < 2^{\aleph_0}$, is perfectly normal.

Metrizable subspaces of $\mathfrak{X}(R)$. Let $S = \bigcup\{S^k: k = 1, 2, \ldots\}$ be a subspace of R such that all non-empty open subsets of any S^k (in the relative topology) have cardinality $|S| = \lambda$, and sets S^k are dense and form a partition of S. Take

$$Y(n) = \{y = \{y^1, \ldots, y^n\}: y^1 \in S^1, \ldots, y^n \in S^n\}$$

and

$$Y(S) = \bigcup\{Y(n): n = 1, 2, \ldots\}.$$

We have $Y(S) \subset \mathfrak{X}(S) \subset \mathfrak{X}(R)$, and thus $Y(S)$ is a subspace of $\mathfrak{X}(R)$.

Let

$$\left\langle y, \frac{1}{m}\right\rangle = \left\langle y, \bigcup\left\{\left(t-\frac{1}{m}, t+\frac{1}{m}\right): t \in y\right\}\right\rangle$$

and let

$$\delta(y) = \min\{1, |t-s|: t, s \in y \text{ and } t \neq s\} \quad \text{for any } y \in X(R).$$

The space $Y(S)$ has a σ-discrete base. Such a base for $Y(S)$ consists of elements of families

$$U_{n,m} = \left\{\left\langle y, \frac{1}{m}\right\rangle \cap Y(S) : y \in Y(n) \text{ and } \delta(y) > \frac{1}{m}\right\} \quad \text{for } m, n = 1, 2, \ldots$$

Each $U_{n,m}$ is discrete in $Y(S)$, since if $z \in Y(S)$ and $|z| \geqslant n$, i.e. if $z = \{z^1, \ldots, z^n, \ldots\}$, $z^1 \in S^1, \ldots, z^n \in S^n, \ldots$, then $\langle\{z^1, \ldots, z^n\}, 1/m\rangle \cap Y(S)$ is the unique set which can belong to $U_{n,m}$ and can intersect any set $\langle z, V\rangle \cap Y(S)$. If $|z| < n$, then the open neighbourhood $\langle z, 1/m\rangle \cap Y(S)$ of z intersects no element of $U_{n,m}$.

Let

$$X(R) = \bigcup\left\{V_{n,m} = \left\{y \in X(R) : |y| = n \text{ and } \delta(y) > \frac{1}{m}\right\}: n, m = 1, 2, \ldots\right\}.$$

Each $V_{n,m}$ is discrete in $X(R)$, since $\langle x, 1/m\rangle \cap V_{n,m}$ can contain only x for any $x \in X(R)$. Thus, by Bing's metrization theorem [3], the subspace $Y(S)$ is a σ-discrete metrizable space of cardinality λ.

Recall that a space is *universal* in a class U if it belongs to U and each space from U can be embedded in it.

THEOREM 1. *Let $\aleph_0 < \lambda \leqslant 2^{\aleph_0}$ and let S be a dense subspace of R such that all non-empty open subsets of S have cardinality λ. Then the space $Y(S) \subset X(S)$ is universal in the class of all σ-discrete metrizable spaces of cardinality (weight) less than or equal to λ.*

Proof. Assume that $Z = \bigcup\{Z_n: n = 1, 2, \ldots\}$ is a σ-discrete metrizable space, where sets Z_n are discrete in Z and form a partition of Z. Let $\{W_z^n: n = 1, 2, \ldots\}$ denote a base, consisting of closed-open sets, at a point $z \in Z$ such that $W_z^n \subset W_z^{n-1}$. Let $\{V_z^0 = W_z^n: z \in Z_1 \text{ and } n = n(z)\}$ be a discrete family which simultaneously separates points of Z_1. Take $V_z^m = W_z^{n+m}$ for $z \in Z_1$.

Assume $\{V_z^n: n = 0, 1, \ldots\}$ have been defined for $z \in Z_1 \cup \ldots \cup Z_{k-1}$. We define $\{V_z^n: n = 0, 1, \ldots\}$ for $z \in Z_k$. Let

$$V_z^0 = W_z^i \subset U_0^z \cap \ldots \cap U_{k-1}^z \quad \text{and} \quad V_z^n = W_z^{i+n},$$

where $\{U_0^z : z \in Z_k\}$ is a discrete family which simultaneously separates points of Z_k. Sets U_m^z $(m = 1, \ldots, k-1)$ are defined in the following manner:

A. If $z \in V_x^1$ for some $x \in Z_m$ (by induction hypotheses, such a point x can be unique for each m provided $z \in V_x^n \setminus V_x^{n-1}$), then let us take $U_m^z = V_x^n \setminus V_x^{n-1}$.

B. If z does not belong to any set V_x^1 for $x \in Z_m$, then — since the family $\{V_x^1 : x \in Z_m\}$ is discrete and sets W_y^n are closed-open — there exists a W_z^n which is disjoint with any element of $\{V_x^1 : x \in Z_m\}$. Take $W_z^n = U_m^z$.

The family $\{V_x^n : n = 1, 2, \ldots\}$ is a base for a point $x \in Z$. Let $P(x) = \{y \in Z : x \in V_y^1$ and $x \neq y\}$ for points $z \in Z$. The sets $P(x)$ are finite and there is at most one point $y \in Z_n$ such that $y \in P(x)$ for each n. If $h, g \in P(y)$ and $|h| < |g|$, then

$$V_g^1 \subset U_{|g|}^h = V_h^i \setminus V_h^{i+1} \subset V_h^1.$$

Therefore, we can order elements of $P(x) = \{y_1, \ldots, y_n\}$ so that

$$\text{(o)} \qquad \emptyset = P(y_1) \subset \ldots \subset P(y_n) \subset P(x).$$

Taking $Y(n)$ instead of Z_n, we can define, similarly as above, bases

$$\left\{ V_y^n = \left\langle y, \frac{1}{i+n} \right\rangle \cap Y(S) : n = 1, 2, \ldots \right\}$$

for points $y \in Y(S)$.

A desired embedding $f: Z \to Y$ will be defined inductively with respect to the cardinality of $P(x)$. Let f be a one-to-one map on the set $\{z \in Z : P(z) = \emptyset\}$ into the set $Y(1)$.

Assume that f has been defined on $\{x \in Z : |P(x)| < n\}$. We define f on $\{x \in Z : |P(x)| = n\}$ in the following way.

If $x \in V_y^i \setminus V_y^{i-1}$, where y is the last element of $P(x)$ in the ordering (o), then we assume that the value $f(x)$ is in the set $Y(n+1) \cap V_{f(y)}^i \setminus V_{f(y)}^{i+1}$ and that $x \neq x'$ implies $f(x) \neq f(x')$. It is possible since S is dense and each S^k contains open non-empty subsets of cardinality λ. Therefore, the set

$$Y(n+1) \cap \left\langle z, \frac{1}{i} \right\rangle \setminus \left\langle z, \frac{1}{i+1} \right\rangle$$

has cardinality λ for any $z \in Y(n)$.

The function $f: Z \to Y$ is one-to-one by definition. To prove that f is an embedding it suffices to verify that $f(V_x^m) = f(Z) \cap V_{f(x)}^m$ for each $x \in Z$ and $m = 1, 2, \ldots$

If $y \in V_x^m$, $x \neq y$, i.e. if $y \in V_x^i \setminus V_x^{i+1}$ for some $i \geqslant m$, then x is a member of $P(y)$. Assume that x has the index k in the ordering (o). We dorff Moore space is metrizable. For $\aleph_1 < 2^{\aleph_0}$, the consistency of positive answers to (*) and (**) with ZFC can be deduced from Baumgartner's results in [2]. However, we do not know any full solutions, i.e. solutions in ZFC.

Some examples of non-metrizable (normal) subspaces of $X(R)$. Let $S \subset R$ be a set with a countable partition $S = \bigcup\{S^k: k = 0, 1, \ldots\}$ into dense subsets such that each intersection of every S^k with any interval has cardinality $|S|$. Let

$$A_i = \{x \subset S^0 \cup S^1 \cup \ldots \cup S^i: |x \cap S^k| \leqslant 1 \text{ for } k = 1, 2, \ldots, i\}$$

and let

$$A(S) = \bigcup\{A_i: i = 0, 1, \ldots\}.$$

The space $A(S)$, as a subspace of $X(R)$, has the following properties:

(i) *If S is uncountable, then each non-empty open subset of $A(S)$ has cellularity $|S|$.*

Indeed, each basic set, i.e. each set of the form $A(S) \cap \langle x, V\rangle$, where x belongs to some A_i, contains exactly $|S|$ open disjoint sets of the form $\langle x \cup \{t\}, V\rangle \cap A(S)$, where $t \in S^{i+1} \cap V$.

(ii) *If Martin's Axiom holds and $|S| < 2^{\aleph_0}$, then $A(S)$ is normal.*

(iii) *Each space $A(S)$ contains a dense metrizable subspace.*

To see this let us consider the family

$$\left\{\left\langle x, \frac{1}{n}\right\rangle: x = \{x_1, \ldots, x_{2k}\} \in A_{2k}, x_1, \ldots, x_k \in S^0, x_{k+1} \in S^1, \ldots, x_{2k} \in S^k \text{ and } n = 1, 2, \ldots\right\} \quad \text{for each } k.$$

This family has cardinality $|S|$. Let $\{V_\gamma: \gamma < |S|\}$ be its well ordering. For each ordinal $\gamma < |S|$ let $y_\gamma = x \cup \{t_\gamma\}$, where $t_\gamma \in V \cap S^{k+1} \setminus \{t_\beta: \beta < \gamma\}$, and let V be such that $V_\gamma = \langle x, V\rangle$. Let G_k be the set of all y_γ and $G = \bigcup\{G_k: k = 1, 2, \ldots\}$. To see that the subspace G has a σ-discrete base, we proceed as in the case of $Y(S)$. Thus G is metrizable by Bing's metrization theorem [3].

The set G is dense in $A(S)$. Indeed, if $\langle y, V\rangle \cap A(S)$ is an arbitrary non-empty basic open set, then it is possible to add to y a finite set $h \subset V$ such that $y \cup h \in G_k$ for some k.

(iv) *If S is uncountable, then $A(S)$ is locally non-metrizable.*

For each basic open set $\langle x, V\rangle \cap A(S)$, consider a subset

$$K = \{x \cup z \in \langle x, V\rangle \cap A(S): z \subset S^0\}.$$

It is uncountable, and hence non-separable. K has cellularity $\aleph_0$. If not, then there exists an uncountable family U of disjoint open sets of the form $K \cap \langle y, 1/n\rangle$, where $|y| = m$.

Let $T = \{y: K \cap \langle y, 1/n\rangle \in U\}$. We can consider the set T as a subset of the m-dimensional Euclidean space R^m, identifying a point $y = \{y_1, \ldots, y_m\}$ such that $y_1 < \ldots < y_m$ with the vector $(y_1, \ldots, y_m)$. Since T is uncountable, there exist v and z in T such that $z \neq v$ and $\varrho(z, v) < 1/n$. Consequently, $z \cup v \in K$ and $z \cup v \in \langle z, 1/n\rangle \cap K \cap \langle v, 1/n\rangle$, but this contradicts the fact that elements of U are disjoint. Thus a non-empty open set of $A(S)$ contains a non-separable subset with cellularity $\aleph_0$, and hence it cannot be metrizable.

Thus we have given examples of subspaces of $X(R)$ which contradict the normal Moore space conjecture [5] under ZFC $+$MA$+\neg$CH. However, one can obtain examples of such pathological (normal) Moore spaces in other ways, e.g. using methods from [1].

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