COLLOQUIUM MATHEMATICUM

VOL. LIV 1987 FASC. 1

EXTREME POINTS OF FAMILIES OF UNIVALENT FUNCTIONS WITH FIXED SECOND COEFFICIENT

BY

O. P. AHUJA (PORT MORESBY) AND H. SILVERMAN (CHARLESTON, SC)

1. Introduction. Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic and univalent in the unit disk $\Delta = \{z\colon |z| < 1\}$. It is known (see [3] and [5]) that a sufficient condition for $z + \sum_{n=2}^{\infty} a_n z^n$ to be in S is that

$$\sum_{n=2}^{\infty} n|a_n| \leqslant 1. \tag{1}$$

For functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geqslant 0,$$

condition (1) is also necessary for univalence because

$$f'(r) = 1 - \sum_{n=2}^{\infty} n a_n r^{n-1} = 0$$

for some r (< 1) if

$$\sum_{n=2}^{\infty} n a_n > 1.$$

Let T denote the subclass of S consisting of functions of the form

$$z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geqslant 0.$$

Denote by $T^*(\alpha)$ and $C(\alpha)$ the families of functions in T that are, respectively, starlike of order α and convex of order α, $0 \leqslant \alpha \leqslant 1$. It was shown in [11] that $f \in T^*(\alpha)$ if and only if

(2)
$$\sum_{n=2}^{\infty} ((n-\alpha)/(1-\alpha))\, a_n \leqslant 1$$

and that $f \in C(\alpha)$ if and only if

$$\sum_{n=2}^{\infty} (n(n-\alpha)/(1-\alpha))\, a_n \leqslant 1.$$

Additional subfamilies of T have been investigated in [1], [6], [10], and [13].

A function

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geqslant 0,$$

is said to be in the *family* $F(\{b_n\})$ if there exists a sequence $\{b_n\}$ of positive real numbers such that

$$\sum_{n=2}^{\infty} b_n a_n \leqslant 1.$$

The family $F(\{b_n\})$ was introduced in [12], where it was observed that $F(\{b_n\})$ is a convex family and is contained in T if and only if $b_n \geqslant n$ for every n, which we will henceforth assume unless otherwise stated. From the definition we see that $a_2 \leqslant 1/b_2$. We now introduce a subfamily of functions in $F(\{b_n\})$ with fixed second coefficient.

A function

$$f(z) = z - (p/b_2) z^2 - \sum_{n=3}^{\infty} a_n z^n \quad (0 \leqslant p \leqslant 1,\ a_n \geqslant 0)$$

is said to be in the *family* $F_p(\{b_n\})$ if there exists a sequence $\{b_n\}$ of positive real numbers such that

$$p + \sum_{n=3}^{\infty} b_n a_n \leqslant 1.$$

In this paper, we obtain the extreme points of $F_p(\{b_n\})$ and use them to establish distortion theorems. The order of starlikeness, radius of convexity, and other extremal properties for functions in $F_p(\{b_n\})$ are also determined.

There have been several papers involving subclasses of S with fixed second coefficient. See, e.g., [2], [4], [7], and [9]. Most of these papers deal with distortion theorems and give (frequently non-sharp) coefficient and radius of convexity bounds. The family $F_p(\{b_n\})$ incorporate numerous subfamilies of T consisting of functions with a fixed second coefficient. The families

$$T_p^*(\alpha) = F_p(\{(n-\alpha)/(1-\alpha)\}) \quad \text{and} \quad C_p(\alpha) = F_p(\{n(n-\alpha)/(1-\alpha)\})$$

were investigated in [14]. The subfamily $D(\alpha)$ of T consisting of functions for which

$$\sum_{n=2}^{\infty} [n^2/((1-\alpha)n+\alpha)]\, a_n \leqslant 1$$

was studied in [13]. For $D_p(\alpha)$ consisting of functions in $D(\alpha)$ for which $a_2 = p(2-\alpha)/4$, we note that

$$D_p(\alpha) = F_p\big(\{(n^2/((1-\alpha)n+\alpha))\}\big).$$

The results obtained here for the family $F_p(\{b_n\})$ give rise to the corresponding results for the classes $T_p^*(\alpha)$, $C_p(\alpha)$, $D_p(\alpha)$, and several other subfamilies of T.

2. Extreme points of $F_p(\{b_n\})$. It is easy to see that $F_p(\{b_n\})$ is closed under convex linear combinations and, therefore, the closed convex hull of $F_p(\{b_n\})$ is simply $F_p(\{b_n\})$. We now determine the extreme points of this class.

THEOREM 1. *The extreme points of $F_p(\{b_n\})$ are given by*

$$f_2(z) = z-(p/b_2)z^2 \quad \textit{and} \quad f_n(z) = z-(p/b_2)z^2-((1-p)/b_n)z^n$$

$(n = 3, 4, \ldots)$.

Proof. It suffices to show that $f \in F_p(\{b_n\})$ if and only if it can be expressed in the form

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z), \quad \text{where } \lambda_n \geqslant 0 \text{ and } \sum_{n=2}^{\infty} \lambda_n = 1.$$

Suppose

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z) = z-(p/b_2)z^2-(1-p)\sum_{n=3}^{\infty} (\lambda_n/b_n)z^n,$$

$\lambda_n \geqslant 0$, $\sum_{n=2}^{\infty} \lambda_n = 1$. Since

$$p+(1-p)\sum_{n=3}^{\infty} b_n(\lambda_n/b_n) = 1-\lambda_2(1-p) \leqslant 1,$$

it follows that $f \in F_p(\{b_n\})$.

Conversely, if

$$f(z) = z-(p/b_2)z^2-\sum_{n=3}^{\infty} a_n z^n$$

is in $F_p(\{b_n\})$, then

$$p+\sum_{n=3}^{\infty} b_n a_n \leqslant 1.$$

Hence $a_n \leqslant (1-p)/b_n$ $(n = 3, 4, \ldots)$. Set $\lambda_n = b_n a_n/(1-p)$ $(n = 3, 4, \ldots)$ and $\lambda_2 = 1-\sum_{n=3}^{\infty} \lambda_n$. Then

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z),$$

and the proof is complete.

Remarks. (i) Since $T = F(\{n\})$, we have found the set of extreme points for all univalent functions of the form

$$z-(p/2)z^2-\sum_{n=3}^{\infty} a_n z^n \quad (a_n \geqslant 0,\ 0 \leqslant p \leqslant 1).$$

(ii) For $b_n = (n-\alpha)/(1-\alpha)$, $0 \leqslant \alpha < 1$, Theorem 1 gives the extreme points of the family $T_p^*(\alpha)$ found in [14].

(iii) For $b_n = n(n-\alpha)/(1-\alpha)$, $0 \leqslant \alpha < 1$, we obtain the extreme points of $C_p(\alpha)$ found in [14].

(iv) For $b_n = n^2/((1-\alpha)n+\alpha)$, we get the extreme points for $D_p(\alpha)$.

3. Distortion properties. To prove our distortion theorems, we will make use of

LEMMA 1. *Set*

$$p_0 = [-(4b_2+b_3-1)+\sqrt{(4b_2+b_3-1)^2+16b_2}]/2,$$

$$r_0 = \frac{-2(1-p)b_2+\sqrt{4(1-p)^2 b_2^2+p^2(1-p)b_3}}{p(1-p)},$$

$$f_3(z) = z-(p/b_2)z^2-((1-p)/b_3)z^3.$$

If $0 \leqslant p < p_0 \leqslant 1$ *and* $r_0 \leqslant r < 1$, *then*

$$(3)\qquad |f_3(re^{i\theta})| \leqslant r\left[\frac{4(1-p)b_2^2+p^2 b_3}{4(1-p)b_2^2} + \frac{p^2 b_3+4(1-p)b_2^2}{2b_2^2 b_3}r^2+\frac{4(1-p)^2 b_2^2+p^2(1-p)b_3}{4b_2^2 b_3^2}r^4\right]^{1/2}.$$

Otherwise, we have

$$(4)\qquad |f_3(re^{i\theta})| \leqslant r+(p/b_2)r^2-((1-p)/b_3)r^3.$$

Equality holds in (3) *for*

$$\theta = \cos^{-1}\left(\frac{p(1-p)r^2-pb_3}{4(1-p)b_2 r}\right)$$

and in (4) *for* $\theta = \pi$.

Proof. One can show that

$$\frac{\partial}{\partial\theta}|f_3|^2 = 0$$

for

$$\theta_1 = 0, \quad \theta_2 = \pi, \quad \theta_3 = \cos^{-1}\left(\frac{p(1-p)r^2 - pb_3}{4(1-p)b_2 r}\right).$$

Since θ_3 is a valid root only when $-1 \leqslant \cos\theta_3 \leqslant 1$, and is indeed then a maximum, we have a third root if and only if $r_0 \leqslant r < 1$ and $0 \leqslant p \leqslant p_0 \leqslant 1$.

THEOREM 2. *If* $f \in F_p(\{b_n\})$, $\{b_n\}$ *an increasing sequence, then*

$$|f(re^{i\theta})| \geqslant f_3(r) \quad (0 \leqslant r < 1,\ 0 \leqslant p \leqslant 1) \tag{5}$$

and

$$|f(re^{i\theta})| \leqslant \max\{\max_\theta |f_3(re^{i\theta})|,\ -f_4(-r)\}. \tag{6}$$

The result is sharp.

Proof. The extremal function must be one of the extreme points of $F_p(\{b_n\})$. But

$$|f_n(re^{i\theta})| \geqslant r - (p/b_2)r^2 - ((1-p)/b_3)r^3 = f_3(r) \quad (n \geqslant 3),$$

which proves (5). To prove (6), note that

$$|f_n(re^{i\theta})| \leqslant r + (p/b_2)r^2 + ((1-p)/b_n)r^n.$$

Since $\{b_n\}$ is increasing, we have

$$|f_n(re^{i\theta})| \leqslant r + (p/b_2)r^2 + ((1-p)/b_4)r^4 = -f_4(-r) \quad (n \geqslant 4).$$

Thus, the only functions whose moduli can exceed $-f_4(-r)$ are cubic polynomials. This completes the proof.

Remark. A comparison of $-f_4(-r)$ and the sharp bounds in Lemma 1 enables us, for any fixed p and r, to determine the precise upper bound for $|f(z)|$, $f \in F_p(\{b_n\})$.

Using methods similar to Lemma 1, one can prove

LEMMA 2. *Set*

$$p_1 = \frac{-(6b_2+b_3-3) + \sqrt{(6b_2+b_3-3)^2 + 72b_2}}{6},$$

$$r_1 = \frac{-3(1-p)b_2 + \sqrt{9(1-p)^2 b_2^2 + 3p^2(1-p)b_3}}{3p(1-p)},$$

$$f_3(z) = z - (p/b_2)z^2 - ((1-p)/b_3)z^3.$$

If $0 \leqslant p \leqslant p_1 \leqslant 1$ *and* $r_1 \leqslant r < 1$, *then*

$$|f_3'(re^{i\theta})| \leqslant \left[\frac{p^2 b_3 + 3(1-p)b_2^2}{3(1-p)b_2^2} + \frac{2p^2 b_3 + 6(1-p)b_2^2}{b_2^2 b_3} r^2 + \frac{9(1-p)^2 b_2^2 + 3p^2(1-p)b_3}{b_2^2 b_3^2} r^4\right]^{1/2}. \tag{7}$$

Otherwise, we have

$$|f_3'(re^{i\theta})| \leqslant 1 + \frac{2p}{b_2} r - \frac{3(1-p)}{b_3} r^2. \tag{8}$$

Equality in (7) *holds when*

$$\cos\theta = \frac{3p(1-p)r^2 - pb_3}{6(1-p)b_2 r},$$

and in (8) *when* $\theta = \pi$.

The next theorem, in conjunction with Lemma 2, enables us for fixed p and r to obtain bounds on f', $f \in F_p(\{b_n\})$. Its proof is similar to that of Theorem 2, and will be omitted.

THEOREM 3. *If* $f \in F_p(\{b_n\})$, $\{b_n\}$ *increasing, then*

$$|f'(re^{i\theta})| \geqslant f_3'(r) \quad (0 \leqslant r < 1,\ 0 \leqslant p \leqslant 1)$$

and

$$|f'(re^{i\theta})| \leqslant \max\{\max_\theta |f_3'(re^{i\theta})|, f_4'(-r)\}.$$

Remarks. (i) For $b_n = (n-\alpha)/(1-\alpha)$ and $b_n = n(n-\alpha)/(1-\alpha)$, Theorems 2 and 3 yield the corresponding distortion results for $T_p^*(\alpha)$ and $C_p(\alpha)$, respectively, obtained in [14].

(ii) For $b_n = n^2/((1-\alpha)n+\alpha)$ the sharp upper and lower bounds for $|f|$ and $|f'|$, $f \in D_p(\alpha)$, follow from the above distortion theorems.

4. Order of starlikeness. We will show that the extremal function for the order of starlikeness for some well-known subfamilies of $F_p(\{b_n\})$ is a cubic polynomial.

THEOREM 4. *If*

$$b_n \geqslant 1 + \frac{(n-1)[b_2(b_3-1) - p(b_3-b_2)]}{2b_2 + p(b_3 - 2b_2)} \tag{9}$$

for $n \geqslant 3$, *then the order of starlikeness of* $F_p(\{b_n\})$ *is*

$$\beta = \beta(p) = \frac{b_2(b_3-3) + p(3b_2 - 2b_3)}{b_2(b_3-1) - p(b_3-b_2)}. \tag{10}$$

The result is sharp, with the extremal function

$$f_3(z) = z - (p/b_2)z^2 - ((1-p)/b_3)z^3.$$

Proof. First note, for β defined by (10), that

$$\frac{p}{b_2}\left(\frac{2-\beta}{1-\beta}\right)+\left(\frac{1-p}{b_3}\right)\left(\frac{3-\beta}{1-\beta}\right)=1.$$

Hence, we see from (2) that $f_3(z)$ is starlike of order β. In view of Theorem 1, we may express an arbitrary function $f(z)\in F_p(\{b_n\})$ as

$$f(z)=z-(p/b_2)z^2-(1-p)\sum_{n=3}^{\infty}(\lambda_n/b_n)z^n,$$

where $\lambda_n \geqslant 0$ and $\sum_{n=3}^{\infty}\lambda_n \leqslant 1$. It therefore suffices to show that

$$\frac{p}{b_2}\left(\frac{2-\beta}{1-\beta}\right)+(1-p)\sum_{n=3}^{\infty}\frac{\lambda_n}{b_n}\left(\frac{n-\beta}{1-\beta}\right)\leqslant 1. \tag{11}$$

But (11) is maximized when $\lambda_3 = 1$ if

$$\frac{1}{b_3}\left(\frac{3-\beta}{1-\beta}\right)\geqslant\frac{1}{b_n}\left(\frac{n-\beta}{1-\beta}\right),$$

which is a consequence of (9). This completes the proof.

Remark. Since $b_n = n(n-\alpha)/(1-\alpha)$ satisfies (9) for $n \geqslant 3$, the inclusion

$$C_p(\alpha)=F_p(\{n(n-\alpha)/(1-\alpha)\})\subset T^*(\beta)$$

when

$$\beta=\frac{6\big(2(2-\alpha)-p(1-\alpha)\big)}{4(2-\alpha)(4-\alpha)-p(1-\alpha)(5-\alpha)},$$

proved in [14], is a consequence of Theorem 4. Further, when $p = 1$, we have $\beta = 2/(3-\alpha)$, the order of starlikeness found in [11]. For $p = 1$, Theorem 4 gives the following result proved in [12].

COROLLARY 1. *If* $b_n \geqslant (b_2-1)(n-1)+1$ *for every* n, *then the order of starlikeness of* $F(\{b_n\})$ *is* $(b_2-2)/(b_2-1)$.

Setting $b_n = n^2/((1-\alpha)n+\alpha)$, we have

COROLLARY 2. *If* $f\in D_p(\alpha)$, *then*

$$f\in T^*\left(\frac{6\alpha(4-p)}{4(6+2\alpha)-p(6-\alpha)}\right).$$

For $p = 1$, Corollary 2 yields the result found in [13].

COROLLARY 3. *If*

$$f(z)=z-\big(p(1-\alpha)/(2-\alpha)\big)z^2-\sum_{n=3}^{\infty}a_n z^n$$

is in $T_p^*(\alpha)$, *then*

$$g(z) = \int_0^z \frac{f(t)}{t}\,dt = z-(p(1-\alpha)/2(2-\alpha))z^2 - \sum_{n=3}^{\infty}(a_n/n)z^n$$

is in

$$T^*\left(\frac{12(2-\alpha)-6p(1-\alpha)}{4(2-\alpha)(4-\alpha)-p(5-\alpha)}\right).$$

Proof. If $b_n = n(n-\alpha)/(1-\alpha)$, then $g \in F_p(\{b_n\})$. Noting that (9) is satisfied, the result follows.

It is known that the Libera transform

$$T(f) = \frac{2}{z}\int_0^z f(t)\,dt$$

preserves convexity, starlikeness, and close-to-convexity [8]. It was shown in [12] that $T(f)$ takes $f \in T^*(\alpha)$ to functions starlike of order $(2+\alpha)/(4-\alpha)$. More generally, we have

COROLLARY 4. *If* $f \in T_p^*(\alpha)$, *then*

$$g(z) = \frac{2}{z}\int_0^z f(t)\,dt$$

is in $T^*(\beta)$ *for*

$$\beta = \frac{3(2-\alpha)(3+\alpha)-p(1-\alpha)(6+\alpha)}{3(2-\alpha)(5-\alpha)-p(1-\alpha)(6-\alpha)}.$$

5. Radii of convexity and starlikeness. We next obtain the radius of convexity for functions in $F_p(\{b_n\})$.

THEOREM 5. *If* $f \in F_p(\{b_n\})$, *then* f *is convex in the disk* $|z| < r_0 = r_0(p)$, *where* $r_0(p)$ *is the largest value of* r *for which*

$$\frac{4p}{b_2}r + \frac{n^2(1-p)r^{n-1}}{b_n} \leqslant 1 \quad (n = 3, 4, \ldots).$$

The result is sharp, with the extremal function being of the form

$$f_n(z) = z-(p/b_2)z^2 - ((1-p)/b_n)z^n \quad \textit{for some } n.$$

Proof. In view of Theorem 1, we may write $f \in F_p(\{b_n\})$ as

$$f(z) = z-(p/b_2)z^2-(1-p)\sum_{n=3}^{\infty}(\lambda_n/b_n)z^n,$$

where $\lambda_n \geqslant 0$ and $\sum_{n=3}^{\infty}\lambda_n \leqslant 1$. It suffices to show that $|zf''/f'| \leqslant 1$ for $|z| \leqslant r_0$.

But

$$\left|\frac{zf''}{f'}\right| \leqslant \frac{(2p/b_2)r+(1-p)\sum_{n=3}^{\infty}(\lambda_n/b_n)n(n-1)r^{n-1}}{1-(2p/b_2)r-(1-p)\sum_{n=3}^{\infty}(\lambda_n/b_n)nr^{n-1}}.$$

Thus $|zf''/f'| \leqslant 1$ if

$$\frac{4p}{b_2}r+(1-p)\sum_{n=3}^{\infty}\frac{\lambda_n n^2}{b_n}r^{n-1} \leqslant 1. \tag{12}$$

For each fixed r, choose the integer $n_0 = n_0(r)$ for which $(n^2/b_n)r^{n-1}$ is maximal. Then

$$\sum_{n=3}^{\infty}(\lambda_n n^2/b_n)r^{n-1} \leqslant (n_0^2/b_{n_0})r^{n_0-1}.$$

We now find the value $r_0 = r_0(p)$ and the corresponding $n_0(r_0)$ so that

$$\frac{4p}{b_2}r_0+\frac{(1-p)n_0^2 r_0^{n_0-1}}{b_{n_0}} = 1.$$

This gives the radius of convexity for $F_p(\{b_n\})$.

As was mentioned earlier, $f \in F_p(\{b_n\})$ is univalent and starlike if $b_n \geqslant n$. In our next theorem, we relax this condition on $\{b_n\}$ and determine the radius of starlikeness.

THEOREM 6. *If $f \in F_p(\{b_n\})$, then f is univalent and starlike in the disk $|z| < r_1 = r_1(p)$, where $r_1(p)$ is the largest value of r for which*

$$\frac{2p}{b_2}r+\frac{(1-p)n}{b_n}r^{n-1} \leqslant 1 \quad (n = 3, 4, \ldots). \tag{13}$$

The result is sharp, with extremal functions of the form

$$f_n(z) = z-(p/b_2)z^2-((1-p)/b_n)z^n.$$

Proof. For

$$f(z) = z-(p/b_2)z^2-(1-p)\sum_{n=3}^{\infty}(\lambda_n/b_n)z^n, \quad \sum_{n=3}^{\infty}\lambda_n \leqslant 1,$$

it suffices to show that $|(zf'/f)-1| \leqslant 1$ for $|z| \leqslant r_1 = r_1(p)$. But

$$\left|\frac{zf'}{f}-1\right| \leqslant \left|\frac{(p/b_2)r+(1-p)\sum_{n=3}^{\infty}(\lambda_n/b_n)(n-1)r^{n-1}}{1-(p/b_2)r-(1-p)\sum_{n=3}^{\infty}(\lambda_n/b_n)r^{n-1}}\right| \leqslant 1$$

if

$$\frac{2p}{b_2}r+(1-p)\sum_{n=3}^{\infty}\frac{\lambda_n n}{b_n}r^{n-1}\leqslant 1. \tag{14}$$

The radius of starlikeness now follows from (14) as the radius of convexity in Theorem 5 followed from (12). To show that this is also the radius of univalence, observe that if equality in (13) holds for

$$f_n(z)=z-(p/b_2)z^2-((1-p)/b_n)z^n,$$

then $f_n'(r_1)=0$.

6. The family $F_{p_{n(i)},N}(\{b_n\})$. Instead of fixing just the second coefficient, one could fix finitely many coefficients. Denote by $F_{p_{n(i)},N}(\{b_n\})$ functions in $F(\{b_n\})$ of the form

$$z-\sum_{i=1}^{N}\frac{p_{n(i)}}{b_{n(i)}}z^{n(i)}-\sum_{n\neq n(i)}a_n z^n,\quad \text{where } \sum_{i=1}^{N}p_{n(i)}=p\leqslant 1.$$

Note, for $N=1$ and $n(1)=2$, that $F_{p_2,1}(\{b_n\})=F_p(\{b_n\})$.

As in Theorem 1, one can prove

THEOREM 7. *The extreme points of* $F_{p_{n(i)},N}(\{b_n\})$ *are*

$$z-\sum_{i=1}^{N}\frac{p_{n(i)}}{b_{n(i)}}z^{n(i)}\quad \textit{and}\quad z-\sum_{i=1}^{N}\frac{p_{n(i)}}{b_{n(i)}}z^{n(i)}-\frac{1-p}{b_n}z^n$$

for $n\neq n(i)$.

The characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done for $F_p(\{b_n\})$. We omit the details.

REFERENCES

[1] O. P. Ahuja and P. K. Jain, *On starlike and convex families with missing coefficients*, Bull. Malaysian Math. Soc. (2) 3 (1980), pp. 95–101.

[2] H. S. Al-Amiri, *On p-close-to-star functions of order α*, Proc. Amer. Math. Soc. 29 (1971), pp. 103–108.

[3] J. W. Alexander, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. 17 (1915), pp. 12–22.

[4] M. Finkelstein, *Growth estimates of convex families*, Proc. Amer. Math. Soc. 18 (1967), pp. 412–418.

[5] A. W. Goodman, *Univalent functions and nonanalytic curves*, ibidem 8 (1957), pp. 598–601.

[6] V. P. Gupta and P. K. Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc. 14 (1976), pp. 409–416.

[7] O. P. Juneja and M. L. Mogra, *Radius of convexity of certain classes of analytic functions*, Pacific J. Math. 78 (1978), pp. 359–368.

[8] R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. 16 (1965), pp. 755–758.

[9] C. P. McCarty, *Functions with real part greater than α*, ibidem 35 (1972), pp. 211–216.

[10] R. Mullins and M. Ziegler, *On a subclass of functions of positive real part with negative coefficients*, Houston J. Math. 8 (2) (1982), pp. 255–271.

[11] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51 (1975), pp. 109–116.

[12] — *Order of starlikeness for multipliers of univalent functions*, J. Math. Anal. Appl. 103 (1) (1984), pp. 48–57.

[13] — and E. M. Silvia, *Convex families of starlike functions*, Houston J. Math. 4 (2) (1978), pp. 263–268.

[14] — *Fixed coefficients for subclasses of starlike functions*, ibidem 7 (1) (1981), pp. 129–136.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PAPUA, NEW GUINEA
UNIVERSITY P.O.
PAPUA, NEW GUINEA

DEPARTMENT OF MATHEMATICS
COLLEGE OF CHARLESTON
CHARLESTON, SC
U. S. A.

Reçu par la Rédaction le 7.10.1983