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EXTENSIONS OF POSITIVE OPERATORS AND EXTREME POINTS. III

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The present work is connected with papers [4], Section 2, and [5], Sections 1 and 2. It is inspired by some results of Bonsall [1] and Portenier [6].

In Section 1 we study a class of positive operators between ordered vector spaces akin to order homomorphisms.

In Section 2 we relate this class to extreme extensions of positive operators and to perfect ideals of Bonsall.

Finally, Section 3 is devoted to a generalization, with a somewhat simpler proof, of a Krein-Milman type theorem due to Bonsall.

Throughout, X and Y stand for ordered real vector spaces with X being directed by its ordering. In Sections 2 and 3, Y is additionally assumed to be an order complete lattice. For a subset $A \subset Y$ we put $A_+ = \{a \in A : a \geqslant 0\}$. Note that Y is directed if and only if $Y = Y_+ - Y_+$.

A vector subspace $J \subset Y$ is called an *order ideal* provided it is order convex, i.e. $[a, b] \subset J$ for each $a, b \in J$. In case Y has an order unit, the maximal order ideals are known to be precisely the kernels of positive functionals on X. This follows by the Kantorovič extension theorem (see, e.g., [4], Theorem 1).

As in [4] and [5], given a vector subspace $M \subset X$ and $T \in L_+(M, Y)$, we put

$$E(T) = \{S \in L_+(X, Y) : S|M = T\}.$$

1. A class of positive operators. Let $S \in L_+(X, Y)$ and denote by I the set of all $x \in X$ such that

(*) $\inf\{S(v) : \pm x \leqslant v \in X\}$

exists and equals 0. It is easy to see that

(i) $S^{-1}(0)_+ \subset I \subset S^{-1}(0)$.

(ii) $S^{-1}(0)$ *and* I *are order ideals in* X.

(iii) $I = I_+ - I_+$ *if and only if the infimum in* (*) *is attained for each* $x \in I$.

To prove (iii) observe first that if $x = x_1 - x_2$, where $x_i \in I_+$, then $\pm x \leqslant x_1 + x_2$ and $S(x_1+x_2) = 0$. Conversely, if $\pm x \leqslant v \in X$ and $S(v) = 0$, then $x = v-(v-x)$ and $v,\ v-x \in I_+$.

Note that, if X is a vector lattice, $I = \{x \in X\colon S(|x|) = 0\}$, whence I is a vector-lattice ideal.

Notation. Suppose Y is a vector lattice and denote by $H(X, Y)$ the set of all $S \in L_+(X, Y)$ such that the infimum in (*) exists for each $x \in X$ and equals $|S(x)|$.

Clearly, in case X is a vector lattice, $S \in H(X, Y)$ if and only if S is a lattice homomorphism. This equivalence can be generalized as follows.

THEOREM 1. *Suppose $S \in L_+(X, Y)$. Then $S \in H(X, Y)$ and the infimum in (*) is attained for each $x \in X$ if and only if the following two conditions are satisfied*:

(1) $S^{-1}(0) = S^{-1}(0)_+ - S^{-1}(0)_+$.

(2) $S(X_+) = \{S(x)_+\colon x \in X\}$ [1].

Proof. The "if" part. In view of (2), given $x \in X$, there exist $v_1, v_2 \in X_+$ such that $S(v_1) = S(x)_+$ and $S(v_2) = S(x)_-$. Then $S(v_1 - v_2) = S(x)$, so that, by (1), there exist $a_1, a_2 \in S^{-1}(0)_+$ with $x = (v_1+a_1)-(v_2+a_2)$. Hence $\pm x \leqslant (v_1+a_1)+(v_2+a_2)$. It follows that

$$|S(x)| \leqslant S(v_1+v_2) = |S(x)|,$$

which yields the assertion.

The "only if" part. As $S \in H(X, Y)$, we have $S^{-1}(0) = I$. Hence (1) follows from (iii). To show (2), given $x \in X$, take $v \in X$ such that $\pm x \leqslant v$ and the infimum in (*) is attained. Then $\pm x \leqslant \frac{1}{2}(v \pm x)$, and so we have

$$|S(x)| = S(x)_+ + S(x)_- \leqslant S\big(\tfrac{1}{2}(v+x)\big) + S\big(\tfrac{1}{2}(v-x)\big) = S(v) = |S(x)|.$$

It follows that $S(x)_+ = S\big(\frac{1}{2}(v+x)\big)$.

Condition (2) obviously implies that

(2′) $S(X_+) = S(X)_+$

and the converse implication holds provided $S(X) = Y$. An operator $S \in L(X, Y)$ which satisfies (1) and (2′) is sometimes called an *order homomorphism* (cf. [7], p. 235). With this terminology we have

COROLLARY 1. *Suppose that $S \in L_+(X, Y)$ and $S(X) = Y$. Then $S \in H(X, Y)$ and the infimum in (*) is attained for each $x \in X$ if and only if S is an order homomorphism.*

Note that condition (2) is automatically satisfied in case $Y = R$ The following example shows that (1) is not implied by the single assumption $S \in H(X, Y)$ even for X Archimedean and $Y = R$.

[1] Here $S(x)_+$ denotes the positive part of $S(x)$.

Example (A. Iwanik). Let X be the vector space of all real polynomials regarded as functions on $[-1, 1]$ with the pointwise ordering. Define $S \in L_+(X, R)$ by $S(x) = x(0)$. It follows from the Weierstrass theorem that $S \in H(X, R)$. Nevertheless, if $v \in X$ and $\pm\xi \leqslant v(\xi)$ for $\xi \in [-1, 1]$, then $v(0) > 0$.

2. Extreme extensions and perfect ideals. Throughout the rest of the paper we assume that Y is an order complete vector lattice. For $S \in L_+(X, Y)$ and $x \in X$ we put

$$S_m(x) = \inf\{S(v)\colon\ \pm x \leqslant v \in X\}.$$

An easy calculation shows that $S_m\colon X \to Y_+$ is sublinear. For each $x \in X$ we have $|S(x)| \leqslant S_m(x)$ and the equality holds if and only if $S \in H(X, Y)$ (see Section 1). In case X is a vector lattice, $S_m(x) = S(|x|)$ for each $x \in X$.

Let M be a vector subspace of X and let $T \in L_+(M, Y)$. A simple modification of the proof of Theorem 3 in [4] yields the following generalization of both that theorem and Théorème 3.5 in [6].

THEOREM 2. *Suppose* $S \in E(T)$. *Then* $S \in \operatorname{extr} E(T)$ *if and only if* $\inf\{S_m(x-z)\colon z \in M\} = 0$ *for each* $x \in X$.

The following is a generalization of Theorem 2 in [5]. The proof is mutatis mutandis the same.

THEOREM 3. *Let* M *be a vector subspace of* X *with* $M = M_+ - M_+$ *and let* $T \in H(M, Y)$. *Then*

(a) $\operatorname{extr} E(T) \subset H(X, Y)$.

(b) *If* $\inf\{|y - T(z)|\colon z \in M\} = 0$ *for each* $y \in Y$, *then*

$$E(T) \cap H(X, Y) \subset \operatorname{extr} E(T).$$

Clearly, the first part of Corollary 2 and Corollary 3 in [5] can be generalized in a similar way.

We say that a (proper) order ideal $J \subset X$ is *u-perfect*, where $u \in X_+$, provided for every $x \in J$ and every positive number ε there exists $x' \in J$ such that $\pm x \leqslant x' + \varepsilon u$. For u being an order unit of X this notion was introduced and studied by Bonsall [1]. In this case we simply call J *perfect* as the notion is independent of the choice of an order unit in X. Theorem 3 yields the following characterization of u-perfect maximal ideals which is a slight generalization of Theorem 1 in [1].

COROLLARY 2. *Suppose* $u \in X_+$, $S \in L_+(X, R)$, *and* $S(u) = 1$. *Then*

$$S \in \operatorname{extr}\{T \in L_+(X, R)\colon\ T(u) = 1\}$$

if and only if $S^{-1}(0)$ *is u-perfect.*

Proof. In view of Theorem 3, it is enough to show that $S \in H(X, R)$ if and only if $S^{-1}(0)$ is u-perfect.

Suppose first $S \in H(X, R)$ and $S(x) = 0$. Then $S_m(x) = 0$, so that, given $\varepsilon > 0$, there exists $v \in X$ such that $\pm x \leqslant v$ and $S(v) < \varepsilon$. It follows that

$$v - S(v)u \in S^{-1}(0) \quad \text{and} \quad \pm x \leqslant \big(v - S(v)u\big) + \varepsilon u.$$

Suppose now $S^{-1}(0)$ is u-perfect and $x \in X$. Then, given $\varepsilon > 0$ there exists $v \in X$ such that

$$\pm\big(x - S(x)u\big) \leqslant v \quad \text{and} \quad S(v) < \varepsilon.$$

Hence $\pm x \leqslant |S(x)|u + v$, and so $S_m(x) < |S(x)| + \varepsilon$.

Applying Corollary 2 and an idea from Section 1 of [5], we shall give new proofs of two known results.

THEOREM 4 ([1], Theorem 3, and [7], Theorem 2.12). *Suppose X has an order unit u and J is a perfect ideal of X. Then J is contained in a perfect maximal ideal. If J is the intersection of the maximal order ideals of X containing J, then it is the intersection of the perfect maximal ideals containing it.*

Proof. Put $X_0 = \mathrm{lin}(J \cup \{u\})$ and $T_0(x + tu) = t$ for $x \in J$. Then $T_0 \in L_+(X_0, R)$ and, in view of Corollary 2,

$$T_0 \in \mathrm{extr}\{T \in L_+(X_0, R)\colon\ T(u) = 1\}.$$

By [5], Theorem 1, there exists $S \in \mathrm{extr}\,E(T_0)$. Hence, in view of [5], Lemma 2, $S \in \mathrm{extr}\{T \in L_+(X, R)\colon\ T(u) = 1\}$. As $J \subset S^{-1}(0)$, an application of Corollary 2 completes the proof of the first assertion.

To prove the second assertion, note that, by assumption, given $x_0 \in X \setminus X_0$, there exists $S_0 \in E(T_0)$ with $S_0(x_0) \neq 0$. Then, modifying slightly the proof of Theorem 1 in [5], we see that we can find $S \in \mathrm{extr}\,E(T_0)$ with $S(x_0) \neq 0$. Hence the same argument as above shows that $S^{-1}(0)$ is perfect.

Note that the first assertion of Theorem 4 implies readily Theorem 2 of [1]. (In [1] those results are obtained in the opposite order.)

3. A Krein-Milman type theorem. Using Theorem 1 of [5] again and an idea of Bonsall, we shall prove a generalization of Theorem 4 in [1] which is itself a generalized form of a theorem originally due to M. Krein and D. Milman. A similar result has been recently announced without proof by Kutateladze ([3], Theorem 1).

THEOREM 5. *Let W be a real vector space, let Y be an order complete vector lattice, and let $P\colon\ W \to Y$ be a sublinear mapping. Then, given $w_0 \in W$, there exists $S \in \mathrm{extr}\{T \in L(W, Y)\colon\ T(w) \leqslant P(w)$ for each $w \in W\}$ such that $S(w_0) = P(w_0)$.*

Proof. Put $X = W \times Y$ and define an ordering in X by putting $(w_1, y_1) \leqslant (w_2, y_2)$ provided $w_1 = w_2$ and $y_1 = y_2$ or $y_2 - y_1 > P(w_1 - w_2)$. With this ordering X becomes an ordered vector space and $\{(0, y): y \in Y\}$ is a majorizing subspace of X.

Define further a positive operator T' on the vector subspace $\{(tw_0, y): t \in R,\ y \in Y\}$ by $T'\big((tw_0, y)\big) = tP(w_0) + y$. Let $S' \in \operatorname{extr} E(T')$ and put $S(w) = S'\big((w, 0)\big)$. We show that $S(w) \leqslant P(w)$ for each $w \in W$. Take $y \in Y$ with $P(w) < y$. Then $(-w, y) > (0, 0)$, so that $S'\big((-w, y)\big) = S(-w) + y \geqslant 0$. It follows that $S(w) \leqslant y$.

Finally, that S is extreme is a consequence of the following observation: each $T_1 \in L(W, Y)$ with $T_1(w) \leqslant P(w)$ for all $w \in W$ and $T_1(w_0) = P(w_0)$ gives rise to an operator $T_1' \in E(T')$ defined by $T_1'\big((w, y)\big) = T_1(w) + y$.

Remarks. 1. Theorem 5 can be proved directly in a similar way as Theorem 1 of [5] (cf. also [2], Theorem 1).

2 (S. S. Kutateladze). Theorem 5 implies directly Theorem 1 of [5]. Indeed, $E(T) = \{S \in L(X, Y): S(x) \leqslant T_e(x)$ for each $x \in X\}$ and T_e is sublinear.

Postscript. 1. As easily seen, the map S_m can be alternatively defined as

$$\inf\{S(z_2) - S(z_1):\ z_2,\ -z_1 \in X_+ \text{ and } z_1 \leqslant x \leqslant z_2\}$$

for $x \in X$. In this form, it has been already considered by S. Dubuc, *Fonctionnelles linéaires positives extrémales*, Comptes Rendus des Séances de l'Académie des Sciences, Série A, 270 (1970), p. 1502-1504, for a related purpose.

2. Results related to Corollary 2 are contained in recent papers by K. R. Goodearl, D. E. Handelman and J. W. Lawrence, *Affine representations of Grothendieck groups and applications to Rickart C^*-algebras and $\aleph_0$-continuous regular rings*, Memoirs of the American Mathematical Society 234 (1980), Section I.4, and by K. R. Goodearl and D. E. Handelman, *Metric completions of partially ordered abelian groups*, Indiana University Mathematical Journal 29 (1980), p. 861-895, Section 3. In the second, a map is considered which, in our setting, coincides with S_m.

3. Theorem 5 is effectively equivalent to the axiom of choice (cf. J. Lembcke, *Two extension theorems effectively equivalent to the axiom of choice*, Bulletin of the London Mathematical Society 11 (1979), p. 285-288).

4. For a generalization of Theorem 5 and other related results see H. Luschgy and W. Thomsen, *Extreme points in the Hahn-Banach-Kantorovič setting*, Pacific Journal of Mathematics, to appear.

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