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THE HAAR MEASURE OF CERTAIN SETS IN THE BOHR GROUP

BY

ROBERT E. DRESSLER AND LOUIS PIGNO (MANHATTAN, KANSAS)

Let $\boldsymbol{T}$ be the circle group and $\boldsymbol{Z}$ the ring of integers. In this paper we compute the Haar measure of the closure in $\overline{\boldsymbol{Z}}$ (the Bohr compactification) of certain arithmetically interesting sequences of integers. For facts about harmonic analysis on LCA groups the reader is referred to [2] and [6].

In what follows $\boldsymbol{D}$ is always a dense subgroup of $\boldsymbol{T}$ (cf. [4], p. 4). We endow $\boldsymbol{D}$ with the discrete topology and let $\hat{\boldsymbol{D}}$ be the compact dual group of $\boldsymbol{D}$. The normalized Haar measure of $\hat{\boldsymbol{D}}$ will be denoted by $m_{\hat{\boldsymbol{D}}}$ and the closure of E in $\hat{\boldsymbol{D}}$ by $\tilde{E}$. We denote by $\bar{E}$ the closure of E in $\overline{\boldsymbol{Z}}$.

THEOREM 1. *Let* $E = \langle n_k\rangle_{k=1}^{\infty} \subset \boldsymbol{Z}$. *Then* $m_{\overline{\boldsymbol{Z}}}(\bar{E}) \leqslant m_{\hat{\boldsymbol{D}}}(\tilde{E})$, *and strict inequality may obtain.*

Proof. Let Λ be the annihilator of $\boldsymbol{D}$ in $\overline{\boldsymbol{Z}}$. Then $\hat{\boldsymbol{D}}$ is isomorphic to $\overline{\boldsymbol{Z}}/\Lambda$. The natural homomorphism from $\overline{\boldsymbol{Z}}$ to $\hat{\boldsymbol{D}}$ is given by $\pi : \gamma \to \gamma|\boldsymbol{D}$ ($\gamma \in \overline{\boldsymbol{Z}}$) and $\ker\pi = \Lambda$. Since $\boldsymbol{D}$ is dense in $\boldsymbol{T}$, we may identify E and $\pi(E)$. Consider any compact set $\tilde{K} \subset \hat{\boldsymbol{D}}$. Put

$$K = \{\gamma \in \overline{\boldsymbol{Z}} : \pi(\gamma) \in \tilde{K}\} = \pi^{-1}(\tilde{K}).$$

We claim that

$$m_{\overline{\boldsymbol{Z}}}(K) = m_{\hat{\boldsymbol{D}}}(\tilde{K}) \quad (\tilde{K} \text{ compact in } \hat{\boldsymbol{D}}). \tag{1}$$

To confirm this let $\xi_{\tilde{K}}$ be the characteristic function of $\tilde{K}$. Then

$$m_{\overline{\boldsymbol{Z}}}(K) = \int_{\overline{\boldsymbol{Z}}} \xi_{\tilde{K}} \circ \pi \, dm_{\overline{\boldsymbol{Z}}}.$$

Using (2) of [6], p. 54 (Weil's formula), it can be shown that

$$m_{\overline{\boldsymbol{Z}}}(K) = \int_{\hat{\boldsymbol{D}}} \int_{\Lambda} \xi_{\tilde{K}}\big(\pi(x+\lambda)\big)\, dm_{\Lambda}(\lambda)\, dm_{\hat{\boldsymbol{D}}}(\tilde{x}), \qquad \text{where } \tilde{x} = \pi(x).$$

Inasmuch as π is a group homomorphism, we gather that

$$m_{\overline{Z}}(K) = \int_{\hat{D}} \xi_{\tilde{K}}(\tilde{x})\, dm_{\hat{D}}(\tilde{x})$$

so $m_{\overline{Z}}(K) = m_{\hat{D}}(\tilde{K})$.

Next, let $E \subset Z$ and consider $\tilde{E}$ and $\bar{E}$. Since

$$\bar{E} \subset \pi^{-1}(\tilde{E}), \tag{2}$$

the inequality of Theorem 1 is a consequence of (1) and (2).

Finally, let $E = \langle 2^n + n\rangle_{n=1}^{\infty}$. It can be shown that $m_{\overline{Z}}(\bar{E}) = 0$. Now, in the group $\hat{D}$ of 2-adic integers (see [2], p. 107), 0 is the unique limit point of $\langle 2^n\rangle_{n=1}^{\infty}$. Since Z^+ is dense in $\hat{D}$, so is E. This means that $m_{\hat{D}}(\tilde{E}) = 1$ and the proof is complete.

Corollary 1. *If $\tilde{E}$ is countable in some $\hat{D}$, then $m_{\overline{Z}}(\bar{E}) = 0$.*

Example. Fix any two sequences $\langle a_n\rangle_{n=1}^{\infty}$ $(a_1 \neq 1)$ and $\langle b_n\rangle_{n=1}^{\infty}$ of natural numbers. Let

$$E_n = \{a_1 a_2 \ldots a_n k\colon\ k = 0, \pm 1, \ldots, \pm b_n\},$$

and put

$$E = \bigcup_{n=1}^{\infty} E_n.$$

Then $m_{\overline{Z}}(\bar{E}) = 0$. In fact, $m_{\overline{Z}}(\overline{E + \ldots + E}) = 0$ for any sum of finitely many E's.

To prove this, take

$$D = \{\exp[2\pi i r / a_1 \ldots a_n]\colon\ r \in Z \text{ and } n \in Z^+\}$$

as our dense subgroup. It follows that 0 is the only accumulation point of $\tilde{E}$; see [5] and [2], p. 107 and 403, in this connection. Thus, $\tilde{E}$ is a countable set, and so Corollary 1 now gives $m_{\overline{Z}}(\bar{E}) = 0$. It also follows that any set of the form $\bar{E} + \ldots + \bar{E} = (E + \ldots + E)^-$ must also be null in $\overline{Z}$, since $\tilde{E} + \ldots + \tilde{E}$ is countable in $\hat{D}$.

To illustrate our example let $A = \{n_1! \ldots n_k!\colon\ n_i, k \in Z^+\}$ and for any natural number m put $A^m = A + \ldots + A$ (m summands). Then $m_{\overline{Z}}(\overline{A_m}) = 0$.

To see this let $a_n = 2$ for all n. For any natural number n, we observe that $2^n l$, where l is odd, can represent only a finite number of integers each of which is a product of factorials. So, let b_n be the largest such l.

Let E be as in the previous example. Then it is possible to prove the following more general result:

Let E^a denote the set of accumulation points of E in $\overline{Z}$ and let $G_p(E^a)$ be the group generated by E^a. Then $m_{\overline{Z}}(G_p(E^a)) = 0$.

To prove this assertion notice that 0 is the only accumulation point of E in $\hat{D}$. Inasmuch as $\hat{D}$ is a quotient group of $\bar{Z}$, we gather that $E^a \subset \Lambda$, where Λ is the annihilator of D in $\bar{Z}$. Suppose that $m_{\bar{Z}}(G_p(E^a)) > 0$. We shall force a contradiction:

It follows from a theorem of Steinhaus that Λ is an open subgroup of $\bar{Z}$. Thus $\hat{D}$ is discrete and compact. The last statement forces $\hat{D}$ to be finite and this contradicts the fact that $\hat{D}$ contains the integers.

THEOREM 2. *The closure in $\bar{Z}$ of the following sets of integers has Haar measure zero*:

(a) *the set $\mathscr{P}$ of prime powers*;

(b) *the set $\mathscr{S}$ of square full integers*;

(c) *the set $\mathscr{D}$ of integers expressible as the sum of two squares*;

(d) *any set $\mathscr{B}$ of all finite sums of the form $\sum_{i=1}^{m}\varepsilon_i n_i$, where $\varepsilon_i \in \{0, 1\}$ and $\langle n_i\rangle_{i=1}^{\infty}$ is a sequence of positive integers such that $n_i | n_{i+1}$ for all i and $n_i+1/n_i > 2$ for infinitely many i.*

Proof. (a) Let p_i be the i-th prime and let $m_k = p_1 \cdot \ldots \cdot p_k$ for $k \in Z^+$. Given $\varepsilon > 0$, it is easy to see that $\varphi(m_k)/m_k < \varepsilon/2$ for k sufficiently large (φ is Euler's function). Now, $\{p^t: t \in Z^+, p \nmid m_k\}$ is a subset of the union of the reduced residue classes $(\operatorname{mod} m_k)$ and, by the above, this union has density less than $\varepsilon/2$.

Choose $r \in Z^+$ so large that

$$\sum_{p\nmid m_k} \frac{1}{p^{r+1}} < \frac{\varepsilon}{2}.$$

Then, all but finitely many elements of the set $\{p^t: t \in Z^+, p \,|\, m_k\}$ are in the set $\bigcup_{p|m_k} p^{r+1}Z$ which has density less than $\varepsilon/2$. The exceptional members are $p_1, \ldots, p_1^r, \ldots, p_k, \ldots, p_k^r$. Thus, with the exception of finitely many members, $\mathscr{P}$ is a subset of a union of residue classes and this union has density less than $\varepsilon/2+\varepsilon/2 = \varepsilon$. Inasmuch as the characteristic function of a residue class is the Fourier-Stieltjes transform of a discrete measure on T, it follows (and this is well known) that the Haar measure of the closure of a residue class is equal to its density. Thus we infer, since $\varepsilon > 0$ is arbitrary, that $m_{\bar{Z}}(\bar{\mathscr{P}}) = 0$.

(b) Recall that a positive integer is square full if all exponents in its canonical factorization are greater than 1.

For each $k \in Z^+$ and for each $1 \leqslant i \leqslant k$, the set $\mathscr{S}$ is a subset of the union of $p_i^2-(p_i-1)$ residue classes $(\operatorname{mod} p_i^2)$; namely those residue classes not generated by $p_i, 2p_i, \ldots, (p_i-1)p_i$. Thus, $\mathscr{S}$ is a subset of a union of residue classes $(\bmod\, p_1^2 \dots p_k^2)$, the density of this union being

$$\prod_{i=1}^{k} \frac{p_i^2-(p_i-1)}{p_i^2}. \tag{3}$$

Since this product diverges to zero as $k \to \infty$, we have $m_{\bar{Z}}(\bar{\mathscr{S}}) = 0$.

COROLLARY 2. *Let* $\mathscr{L} = \{n^k : n \in Z^+, k \geqslant 2\}$. *Then* $m_{\bar{Z}}(\bar{\mathscr{L}}) = 0$.

Proof. All elements of $\mathscr{L}$, except 1, are elements of $\mathscr{S}$.

(c) It is well known that an integer d is in $\mathscr{D}$ if and only if every prime of the form $4n+3$ which divides d appears with even exponent in the canonical factorization of d. Let p_i be the i-th prime of the form $4n+3$. The proof now follows as in (b) and we infer that $\mathscr{D}$ is a subset of a union of residue classes $(\bmod\, p_1^2 \dots p_k^2)$, the density of this union being equal to product (3). This product diverges to zero as $k \to \infty$ by Dirichlet's theorem, so $m_{\bar{Z}}(\bar{\mathscr{D}}) = 0$.

(d) Choose $\varepsilon > 0$. There is a positive integer N such that $2^{N-1}/n_N < \varepsilon$. Now $\mathscr{B}$ is a subset of the union of 2^{N-1} residue classes $(\bmod\, n_N)$; namely those residue classes generated by all integers of the form $\sum_{i=1}^{N-1} \varepsilon_i n_i$, $\varepsilon_i \in \{0, 1\}$. This union has density $2^{N-1}/n_N$ which is less than ε. Since $\varepsilon > 0$ is arbitrary, we may conclude that $m_{\bar{Z}}(\bar{\mathscr{B}}) = 0$.

Comments. (i) In contrast to part (b) of Theorem 2, the Haar measure of the closure in $\bar{Z}$ of the set $\mathscr{G}$ of integers expressible as a sum of three squares is at least $5/16$. This follows from the fact that an integer is expressible as a sum of three squares if and only if it is not of the form $4^j(8k+7)$, $j, k \geqslant 0$, which, in turn, implies that $m_{\bar{Z}}(\bar{\mathscr{G}} \cup -\bar{\mathscr{G}}) \geqslant 5/8$. It then follows that $m_{\bar{Z}}(\bar{\mathscr{G}}) \geqslant 5/16$.

(ii) In contrast to part (c) of Theorem 2, the Haar measure of the closure in $\bar{Z}$ of the set of non-square free integers (positive and negative) is $1-6/\pi^2$. This follows from the fact that $\prod(1-p^{-2}) = 6/\pi^2$, where the product is taken over all primes p. Thus the Haar measure of the closure in $\bar{Z}$ of the set of positive non-square free integers is at least $1/2-3/\pi^2$.

(iii) All the sets $E + \dots + E$ in the Example are Rosenthal (see [5]). It would be interesting to know if the closure of every Rosenthal set in $\bar{Z}$ has Haar measure zero. This question remains open even for Sidon sets. One positive result is the following: If H is a Hadamard set, then $m_{\bar{Z}}(\bar{H}) = 0$ (see [3]).

(iv) For related work, the reader is referred to [1].

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KANSAS STATE UNIVERSITY
MANHATTAN, KANSAS

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