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EVERY LATTICE IS EMBEDDABLE IN THE LATTICE OF T_1-TOPOLOGIES

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In this note we answer one of the questions posed by R. Duda and stated as P 749. As usual, $\Sigma(X)$ denotes the lattice of topologies on a fixed set X, and $\Lambda(X)$ denotes the lattice of T_1-topologies on X. R. E. Larson has already solved the first part of Duda's problem in the affirmative:

Every lattice can be realized as a sublattice of $\Sigma(X)$ for some set X (see P 749, R 1, p. 161).

Our solution depends on Larson's result and on the following theorem:

THEOREM 1. *For any infinite X, $\Sigma(X)$ is embeddable in $\Lambda(X)$.*

Proof. It suffices to prove that, given an infinite set Y, there exists a set X such that $|X| = |Y|$ and $\Sigma(Y)$ is embeddable in $\Lambda(X)$. It is convenient to break the proof into steps.

(1) Let Y be an infinite set, and let $\{J_y | y \in Y\}$ be a family of pairwise disjoint sets, each of countably infinite cardinality, such that $J_y \cap Y = \{y\}$ for each $y \in Y$. Let $X = \bigcup \{J_y | y \in Y\}$, and let $\mathscr{D}$ be the T_1-topology on X whose open sets are

$$\{\emptyset\} \cup \{A \subseteq X \,|\, (X-A) \cap J_y \text{ is finite for each } y \in Y\}.$$

For $B \subseteq Y$, let $B^* = \bigcup \{J_y | y \in B\}$. Define a function $f: \Sigma(Y) \to \Lambda(X)$ by $f(\mathscr{I}) = \mathscr{D} \vee \{G^* | G \in \mathscr{I}\}$. We claim that f is a 1-1 homomorphism of $\Sigma(Y)$ into $\Lambda(X)$. The following fact is useful in establishing this:

(2) For $\mathscr{I} \in \Sigma(Y)$, $f(\mathscr{I}) = \{G^* \cap D | G \in \mathscr{I}$ and $D \in \mathscr{D}\}$. To see this set $\mathscr{A}$ equal to the latter set. Because $\mathscr{D} \cup \{G^* | G \in \mathscr{I}\} \subseteq \mathscr{A} \subseteq f(\mathscr{I})$, it suffices to prove that $\mathscr{A}$ is a topology on X. Note that $\emptyset$, $X \in \mathscr{A}$, and that $\mathscr{A}$ is closed under finite intersections. Let $G_i \in \mathscr{I}$ and $D_i \in \mathscr{D}$ for each $i \in I$. Then

$$\mathscr{O} = \bigcup \{G_i^* \cap D_i | i \in I\} = G^* \cap D,$$

where

$$G = \bigcup \{G_i | i \in I\}$$

and

$$D = (\bigcup \{J_y \cap \mathcal{O} \mid y \in G\}) \cup (\bigcup \{J_z \mid z \in Y - G\}).$$

Then $D \in \mathcal{D}$ and $\mathcal{A}$ is a topology on X.

(3) $f(\mathcal{S} \vee \mathcal{J}) = f(\mathcal{S}) \vee f(\mathcal{J})$ for all $\mathcal{S}, \mathcal{J} \in \Sigma(Y)$. Because f is monotonic, the relation $\supseteq$ holds. Let $\mathcal{O} \in f(\mathcal{S} \vee \mathcal{J})$. By (2),

$$\mathcal{O} = \bigcup \{(G_i^* \cap D_i) \cap (H_i^* \cap D_i) \mid i \in I\},$$

where $G_i \in \mathcal{S}$, $H_i \in \mathcal{J}$, and $D_i \in \mathcal{D}$ for all $i \in I$. It follows immediately that $\mathcal{O} \in f(\mathcal{S}) \vee f(\mathcal{J})$.

(4) $f(\mathcal{S} \wedge \mathcal{J}) = f(\mathcal{S}) \wedge f(\mathcal{J})$ for all $\mathcal{S}, \mathcal{J} \in \Sigma(Y)$. By the monotonicity of f, the relation $\subseteq$ holds. Let $\mathcal{O} \in f(\mathcal{S}) \wedge f(\mathcal{J})$. By (2),

$$\mathcal{O} = G^* \cap D = H^* \cap E \quad \text{for some } G \in \mathcal{S}, H \in \mathcal{J}, \text{ and } D, E \in \mathcal{D}.$$

For any $y \in G$, $J_y \cap H^* \neq \emptyset$, so $G \subseteq H$. By symmetry, $G = H$ and, therefore,

$$\mathcal{O} = G^* \cap (D \cap E) = H^* \cap (D \cap E) \in f(\mathcal{S} \wedge \mathcal{J}).$$

(5) f is 1-1. For suppose $\mathcal{S}, \mathcal{J} \in \Sigma(Y)$ are distinct; say $G \in \mathcal{S} - \mathcal{J}$. Then $G^* \in f(\mathcal{S}) - f(\mathcal{J})$; otherwise $G^* = H^* \cap D$ for some $H \in \mathcal{J}$ and some $D \in \mathcal{D}$. But then $G = H \in \mathcal{J}$, a contradiction.

This completes the proof.

COROLLARY. *Given a lattice $\mathfrak{A}$, there exists an X such that $\mathfrak{A}$ is embeddable in $\Lambda(X)$.*

Proof. By Larson's result, there exists an X such that $\mathfrak{A}$ is embeddable in $\Sigma(X)$. Since $\Sigma(X)$ is embeddable in $\Sigma(Z)$ for any Z containing X, we may assume X is infinite. By Theorem 1, the statement follows.