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AN AXIOMATIC CHARACTERIZATION OF BOOLEAN-VALUED MODELS FOR SET THEORY

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1. Solovay and Tennenbaum gave in [2] an axiomatic characterization of the models $V_s^{\mathscr{B}}$. As far as we know, the proof of this characterization was never published. The aim of this paper is to give a proof of that characterization.

In this paper $\mathscr{B}$ will denote a fixed complete Boolean algebra. We will denote by $+(\sum)$ the join of two (respectively, infinitely many) elements of Boolean algebra, and by $-$ the Boolean complement. Analogously, $\cdot(\prod)$ stands for meets (sometimes the symbol $\cdot$ will be omitted). $\mathscr{B}(X)$ will denote a class of all functions such that their domains are some sets included in X, and their values belong to $\mathscr{B}$ (X can be a class).

2. By definition, the model $V^{\mathscr{B}}$ is a triple $\langle V^{\mathscr{B}}, \|\cdot = \cdot\|, \|\cdot\in\cdot\|\rangle$, where $V^{\mathscr{B}} = \mathscr{B}(V^{\mathscr{B}})$ (this definition is correct by the axiom of foundation), and Boolean-valued relations $\|\cdot = \cdot\|$ and $\|\cdot\in\cdot\|$ are defined simultaneously by recursion as follows:

$$\|u = v\| = \prod_{x\in \mathrm{dom}(u)} \big(u(x) \Rightarrow \|x\in v\|\big)\cdot \prod_{y\in \mathrm{dom}(v)} \big(v(y) \Rightarrow \|y\in u\|\big),$$

$$\|u\in v\| = \sum_{y\in \mathrm{dom}(v)} \big(v(y)\|u = y\|\big).$$

If φ is a formula of language of model $V^{\mathscr{B}}$, we define in the standard way the Boolean truth value $\|\varphi\|$ which is in $\mathscr{B}$. We say that the sentence φ is *$\mathscr{B}$-valid in the model $V^{\mathscr{B}}$* if $\|\varphi\| = 1$.

Scott in [1] studied models $V^{\mathscr{B}}$ where, in particular, there was proved that all axioms of Zermelo-Frankel set theory and the axiom of choice are $\mathscr{B}$-valid in the model $V^{\mathscr{B}}$.

A function $u\in V^{\mathscr{B}}$ is called *extensional* if, for every $x, y\in \mathrm{dom}(u)$,

$$u(x)\|x = y\| \leqslant u(y).$$

Analogously, we define a notion of extensional function for some functions from a subset (or a subclass) of $V_s^{\mathscr{B}}$ (or R) in $\mathscr{B}$ (see Sections 3 and 4). By $V_{\mathrm{ext}}^{\mathscr{B}}$ we denote the class of all functions that are hereditarily extensional in $V^{\mathscr{B}}$, i. e.

$$V_{\mathrm{ext}}^{\mathscr{B}} = \{u \in \mathscr{B}(V_{\mathrm{ext}}^{\mathscr{B}}): u \text{ is an extensional function}\}.$$

We can prove that for any function $u \in V^{\mathscr{B}}$ there exists a function $v \in V_{\mathrm{ext}}^{\mathscr{B}}$ such that $\|u = v\| = \mathbf{1}$.

3. The model $V_s^{\mathscr{B}}$ is a sequence $\langle V_s^{\mathscr{B}}, \|\cdot = \cdot\|_s, \|\cdot \in \cdot\|_s\rangle$, where $V_s^{\mathscr{B}}$ denotes the class $\{[u]: u \in V^{\mathscr{B}}\}$ and $[u] = \{v \in V^{\mathscr{B}}: \|u = v\| = \mathbf{1}$, and if $\|w = u\| = \mathbf{1}$, then rank $(w) \geqslant$ rank $(v)\}$.

The Boolean-valued relations are defined by the formulas

$$\|[u] = [v]\|_s = \|u = v\|,$$
$$\|[u] \in [v]\|_s = \|u \in v\| \quad \text{for every } u, v \in V^{\mathscr{B}}.$$

LEMMA 1.

(i) *$V_s^{\mathscr{B}}$ is a $\mathscr{B}$-valued model with predicates $=$ and $\in$ such that usual equality axioms are $\mathscr{B}$-valid in $V_s^{\mathscr{B}}$.*

(ii) *For every $x, y \in V_s^{\mathscr{B}}$, if $\|x = y\|_s = \mathbf{1}$, then $x = y$.*

(iii) *The axioms of extensionality and foundation are $\mathscr{B}$-valid in $V_s^{\mathscr{B}}$.*

(iv) *There is a function π defined on a class $\mathscr{B}(V_s^{\mathscr{B}})$ onto $V_s^{\mathscr{B}}$ such that*

$$\|x \in \pi(u)\|_s = \sum_{y \in \mathrm{dom}(u)} \big(u(y) \, \|x = y\|_s\big) \tag{1}$$

for every $x \in V_s^{\mathscr{B}}$ and $u \in \mathscr{B}(V_s^{\mathscr{B}})$.

Proof of (iv). Notice that if for some function $u \in \mathscr{B}(V_s^{\mathscr{B}})$ there exists $y \in V_s^{\mathscr{B}}$ such that

$$\|x \in y\|_s = \sum_{t \in \mathrm{dom}(u)} \big(u(t) \, \|t = x\|_s\big),$$

then, by (ii) and (iii), y is unique.

Let $u \in \mathscr{B}^d$ for some set $d \subseteq V_s^{\mathscr{B}}$. From every set $x \in d$ we choose an element v_x. Clearly, $[v_x] = x$. We can write $\pi(u) = [v]$, where $v \in \mathscr{B}^{\{v_x : x \in d\}}$ and $v(v_x) = u(x)$. $[v]$ has property (1), because

$$\|[w] \in [v]\|_s = \|w \in v\| = \sum_{x \in d} \big(v(v_x) \, \|v_x = w\|\big)$$
$$= \sum_{x \in d} \big(u(x) \, \|[v_x] = [w]\|_s\big) = \sum_{x \in d} \big(u(x) \, \|x = [w]\|_s\big).$$

We now prove that for every $x \in V_s^{\mathscr{B}}$ there exists $u \in \mathscr{B}(V_s^{\mathscr{B}})$ such that $\pi(u) = x$. Let $x \in V_s^{\mathscr{B}}$. Then there is an extensional function $v \in V^{\mathscr{B}}$

such that $[v] = x$. Consider a function $u \in \mathcal{B}^{\{[w]:w \in \mathrm{dom}(v)\}}$ such that $u([w]) = v(w)$. We have $\pi(u) = [v]$, because

$$\|[t] \in [v]\|_s = \|t \in v\| = \sum_{w \in \mathrm{dom}(v)} \big(v(w) \|t = w\|\big)$$

$$= \sum_{w \in \mathrm{dom}(v)} \big(u([w]) \|[t] = [w]\|_s\big) = \sum_{z \in \mathrm{dom}(u)} \big(u(z) \|[t] = z\|_s\big).$$

4. Let X be a class or a set. A class (a set) $X \subseteq V_s^{\mathcal{B}}$ ($X \subseteq V^{\mathcal{B}}$ or $X \subseteq R$) is called *complete* if, for every extensional function F from X into $\mathcal{B}$, there is $y \in X$ such that

$$F(y) = \sum_{t \in X} F(t).$$

Let $\langle R, \|\cdot = \cdot\|_R, \|\cdot \in \cdot\|_R\rangle$ be a Boolean-valued model which satisfies (i)-(iv), i. e. if in Lemma 1 we replace $V_s^{\mathcal{B}}$ by R, $\|\cdot = \cdot\|_s$ and $\|\cdot \in \cdot\|_s$ by $\|\cdot = \cdot\|_R$ and $\|\cdot \in \cdot\|_R$, respectively, and π by π_R, then conditions (i)-(iv) are satisfied.

LEMMA 2. *Let X be a class or a set included in R. Then the class $\pi_R(\mathcal{B}(X))$ is complete.*

LEMMA 3. *For every $x \in R$ there is a function v defined on a complete set $d \subseteq R$ such that $v(t) = \|t \neq 0\|_R$ for $t \in d$, where $0 = \pi_R(\emptyset)$, and $\pi_R(v) = x$.*

Lemmas 2 and 3 are simple generalizations of analogous lemmas from the Scott's paper [1].

5. THEOREM. *The model $V_s^{\mathcal{B}}$ is the unique one (up to an isomorphism) which satisfies conditions* (i)-(iv) *from Lemma* 1.

Proof. We will prove that the model R is isomorphic with $V_s^{\mathcal{B}}$. Let S be the smallest class such that $S = \pi_R(\mathcal{B}(S))$. Clearly, $S \subseteq R$. Consider a function $f: V_{\mathrm{ext}}^{\mathcal{B}} \to R$ which is characterized by the formula

$$f(u) = \pi_R\big(\{\langle f(x), u(x)\rangle : x \in \mathrm{dom}(u)\}\big).$$

The function f has the following properties:

1. $\{\langle f(x), u(x)\rangle : x \in \mathrm{dom}(u)\}$ is a function if $u \in V_{\mathrm{ext}}^{\mathcal{B}}$, i. e. the definition of f is correct.

2. $f(V_{\mathrm{ext}}^{\mathcal{B}}) \subseteq S$.

3. $\|f(u) = f(v)\|_R = \|u = v\|$ for $u, v \in V_{\mathrm{ext}}^{\mathcal{B}}$.

These properties will be proved simultaneously by recursion. Assume that for every $x \in \mathrm{dom}(u)$ and $y \in \mathrm{dom}(v)$ properties 1, 2 and 3 are valid ($u, v \in V_{\mathrm{ext}}^{\mathcal{B}}$).

1. If $x_1, x_2 \in \mathrm{dom}(u), f(x_1) = f(x_2)$, and u is an extensional function, then $\mathbf{1} = \|f(x_1) = f(x_2)\|_R = \|x_1 = x_2\| \leqslant u(x_1) \Leftrightarrow u(x_2)$, i. e. $u(x_1) = u(x_2)$.

2. If $f(\operatorname{dom}(u)) \subseteq S$, then, by the definition of S,

$$f(u) = \pi_R(\{\langle f(x), u(x)\rangle : x \in \operatorname{dom}(u)\}) \in S.$$

(By 1 this formula makes sense.)

3. The axiom of extensionality is $\mathscr{B}$-valid in R and π_R has property (iv). Hence

$$\begin{aligned}
&\|f(u) = f(v)\|_R \\
&= \prod_{x \in \operatorname{dom}(u)} \big(u(x) \Rightarrow \|f(x) \in f(v)\|_R\big) \cdot \prod_{y \in \operatorname{dom}(v)} \big(v(y) \Rightarrow \|f(y) \in f(u)\|_R\big) \\
&= \prod_{x \in \operatorname{dom}(u)} \Big(u(x) \Rightarrow \sum_{y \in \operatorname{dom}(v)} \big(v(y)\,\|f(x) = f(y)\|_R\big)\Big) \cdot \\
&\qquad\qquad \cdot \prod_{y \in \operatorname{dom}(v)} \Big(v(y) \Rightarrow \sum_{x \in \operatorname{dom}(u)} \big(u(x)\,\|f(x) = f(y)\|_R\big)\Big) \\
&= \prod_{x \in \operatorname{dom}(u)} \Big(u(x) \Rightarrow \sum_{y \in \operatorname{dom}(v)} \big(v(y)\,\|x = y\|\big)\Big) \cdot \\
&\qquad\qquad \cdot \prod_{y \in \operatorname{dom}(v)} \Big(v(y) \Rightarrow \sum_{x \in \operatorname{dom}(u)} \big(u(x)\,\|x = y\|\big)\Big) \\
&= \prod_{x \in \operatorname{dom}(u)} \big(u(x) \Rightarrow \|x \in v\|\big) \cdot \prod_{y \in \operatorname{dom}(v)} \big(v(y) \Rightarrow \|y \in u\|\big) = \|u = v\|.
\end{aligned}$$

4. $\|f(u) \in f(v)\|_R = \|u \in v\|$.

5. $f(V^{\mathscr{B}}_{\mathrm{ext}}) = S$.

It only suffices to prove that $\pi_R(u) \in f(V^{\mathscr{B}}_{\mathrm{ext}})$ for every extensional function $u \in \mathscr{B}(f(V^{\mathscr{B}}_{\mathrm{ext}}))$. If $\operatorname{dom}(u) \subseteq f(V^{\mathscr{B}}_{\mathrm{ext}})$, then there exists a set $X \subseteq V^{\mathscr{B}}_{\mathrm{ext}}$ such that $f(X) = \operatorname{dom}(u)$. Take the function $v = \{\langle x, u(f(x))\rangle : x \in X\}$. Clearly, $v \in V^{\mathscr{B}}_{\mathrm{ext}}$. Let $y \in R$. Then

$$\begin{aligned}
\|y \in f(v)\|_R &= \sum_{x \in X} \big(u(f(x))\,\|f(x) = y\|_R\big) = \sum_{z \in f(X)} \big(u(z)\,\|z = y\|_R\big) \\
&= \sum_{z \in \operatorname{dom}(u)} \big(u(z)\,\|z = y\|_R\big) = \|y \in \pi_R(u)\|_R,
\end{aligned}$$

i. e. $f(v) = \pi_R(u)$. In that case $\pi_R(u) \in f(V^{\mathscr{B}}_{\mathrm{ext}})$.

Now, we can easily prove that mapping h which is defined by the formula $h([u]) = f(u)$ for every $u \in V^{\mathscr{B}}_{\mathrm{ext}}$ is an isomorphism of models $V^{\mathscr{B}}_s$ and S.

We will show that $S = R$. Let

$$\mu(x) = \sum_{t \in S} \|t = x\|_R \quad \text{for } x \in R.$$

We will prove that $x \in S$ iff $\mu(x) = 1$. Clearly, $\mu(x) = 1$ if $x \in S$. Assume that $\mu(x) = 1$. By Lemma 2, S is a complete class. Since the function $\|\cdot = x\|_R$ is extensional, there is $x_0 \in S$ such that

$$\|x_0 = x\|_R = \sum_{t \in S} \|t = x\|_R = \mu(x) = 1.$$

In that case $x = x_0 \in S$.

Assume that $x \in R \setminus S$. Clearly, $\mu(x) < 1$. Since the function $-\mu(t)$ is extensional, there is x_0 such that

$$\mu(x_0) = \prod_{t \in R} \mu(t).$$

By Lemma 3, there is a complete set $d \subseteq R$ such that

$$x_0 = \pi_R(d \times \{\|x_0 \neq 0\|_R\}).$$

Let $x_1 \in d$ and let

$$\mu(x_1) = \prod_{t \in d} \mu(t).$$

x_1 has the following properties:

6. $\|x_1 \in x_0\|_R = \sum_{t \in d} (\|x_0 \neq 0\|_R \|x_1 = t\|_R) = \|x_0 \neq 0\|_R$,

7. $\mu(x_1) = \mu(x_0)$.

Proof of 7. From the definition of x_0 it follows that $\mu(x_1) \geqslant \mu(x_0)$. Assume that

$$\mu(x_1) = \prod_{t \in d} \mu(t) > \mu(x_0).$$

We will get a contradiction. For every $t \in d$ we choose $z_t \in S$ such that $\mu(t) = \|t = z_t\|_R$. Let

$$x' = \pi_R(\{\langle z_t, \|x_0 \neq 0\|_R\rangle; t \in d\}) \in S.$$

Then

$$\begin{aligned} &\|x_0 = x'\|_R \\ &= \prod_{t \in d} \Big(\|x_0 \neq 0\|_R \Rightarrow \sum_{z \in d} (\|x_0 \neq 0\|_R \|z_z = t\|_R)\Big) \cdot \\ &\qquad\qquad \cdot \prod_{t \in d} \Big(\|x_0 \neq 0\|_R \Rightarrow \sum_{z \in d} (\|x_0 \neq 0\|_R \|z_t = z\|_R)\Big) \\ &\geqslant \prod_{t \in d} (\|x_0 \neq 0\|_R \Rightarrow \|z_t = t\|_R) \geqslant \prod_{t \in d} \mu(t) = \mu(x_1) > \mu(x_0). \end{aligned}$$

On the other hand, $\|x' = x_0\|_R \leqslant \mu(x_0)$. A contradiction.

By recursion we can prove the existence of a sequence x_n such that, for every $n \in \omega$,

8. $\|x_{n+1} \in x_n\|_R = \|x_n \neq 0\|_R$,

9. $\mu(x_n) = \mu(x_0)$.

Since $0 = \pi_R(\emptyset) \in S$, $-\mu(x_0) = -\mu(x_n) \leqslant \|x_n \neq 0\|_R = \|x_{n+1} \in x_n\|_R$. Consider

$$x = \pi_R(\{\langle x_n, 1\rangle; n \in \omega\}) \in R.$$

For that element we have

$$\|\forall y \in x \, \exists t \in x \; t \in y\|_R = \prod_{n \in \omega} \sum_{m \in \omega} \|x_m \in x_n\|_R$$

$$\geqslant \prod_{n \in \omega} \|x_{n+1} \in x_n\|_R \geqslant -\mu(x_0) > 0.$$

Hence, the axiom of foundation is not $\mathscr{B}$-valid in R which contradicts our assumption (iii).

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