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DEVIL'S STAIRCASES, RAMPS, HUMPS AND ROLLER COASTERS

BY

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1. Introduction. All analysts are familiar with the devil's staircase defined as the unique continuous function $g : [0,1] \to \mathbf{R}$ with

$$g\Big(2\sum_{i=1}^{\infty} \alpha_i 3^{-i}\Big) = \sum_{i=1}^{\infty} \alpha_i 2^{-i} \quad \text{for } \alpha_1, \alpha_2, \ldots \in \{0,1\}.$$

See e.g. [9], p. 99. Let us summarise some of its properties.

LEMMA 1.1. *There is a continuous function $g : [0,1] \to \mathbf{R}$ together with a collection $\mathcal{G}$ of disjoint closed subintervals of $[0,1]$ having the following properties.*

(i) $g(0) = 0$, $g(1) = 1$.

(ii) *g is increasing.*

(iii) *If $x \in (0,1]$ but $x \notin \bigcup_{[a,b]\in\mathcal{G}}[a,b]$ then given any $\delta > 0$ we can find $[a,b] \in \mathcal{G}$ such that $\delta > 2(b-a) \geq x - a \geq 0$.*

(iii)$'$ *If $x \in [0,1)$ but $x \notin \bigcup_{[a,b]\in\mathcal{G}}[a,b]$ then given any $\delta > 0$ we can find $[a,b] \in \mathcal{G}$ such that $\delta > 2(b-a) \geq b - x \geq 0$.*

(iv) *g is constant on each $[a,b] \in \mathcal{G}$.*

Proof. Easy. We take

$$\mathcal{G} = \Big\{ \Big[2\sum_{i=1}^{N-1} \alpha_i 3^{-i} + 3^{-N},\ 2\sum_{i=1}^{N-1} \alpha_i 3^{-i} + 2\cdot 3^{-N}\Big] : \alpha_1, \ldots, \alpha_{N-1} \in \{0,1\},\ N \geq 1\Big\}. \blacksquare$$

We now form a "devil's hump" $f : [0,1] \to \mathbf{R}$ by setting

$$f(t) = g(2t) \quad \text{for } 0 \leq t \leq 1/2, \qquad f(t) = f(1-t) \quad \text{for } 1/2 \leq t \leq 1.$$

We define an associated $\mathcal{F}$ in the obvious way by

$$\mathcal{F} = \bigcup \{ [a/2, b/2], [1 - b/2, 1 - a/2] : [a,b] \in \mathcal{G}\},$$

and observe that f is constant on each $[a,b] \in \mathcal{F}$.

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Let us write χ_E for the characteristic function of the set E (so $\chi_E(t) = 1$ if $t \in E$, $\chi_E(t) = 0$ otherwise). We now produce iteratively a sequence of functions $f_n : [0,1] \to \mathbf{R}$ and sets $\mathcal{F}(n)$ by the rule $f_0 = 0$, $\mathcal{F}(0) = [0,1]$ and

$$f_n(x) = f(x) + \sum_{[a,b]\in\mathcal{F}} (b-a)\chi_{[a,b]}(x) f_{n-1}((x-a)/(b-a)),$$

$$\mathcal{F}(n) = \{\, [a' + a(b'-a'), a' + b(b'-a')] : [a,b] \in \mathcal{F},\ [a',b'] \in \mathcal{F}(n-1) \,\}.$$

Observe that

$$f_{n+m}(x) = f_m(x) + \sum_{[a,b]\in\mathcal{F}(m)} (b-a)\chi_{[a,b]}(x) f_n((x-a)/(b-a)).$$

Since this is the construction which forms the main theme of this paper the reader is strongly urged to draw a few diagrams at this point.

THEOREM 1.2. (i) *The sequence f_n converges uniformly on $[0,1]$ to a continuous function F satisfying the relation*

$$F(x) = f(x) + \sum_{[a,b]\in\mathcal{F}(m)} (b-a)\chi_{[a,b]}(x) F((x-a)/(b-a))$$

for all $x \in [0,1]$. We have $F(0) = F(1) = 0$, $F(1/2) = 1$.

(ii) *$F(x) = F(a) + (b-a)F((x-a)/(b-a))$ for all $x \in [a,b]$, $[a,b] \in \mathcal{F}(m)$.*

(iii) *If $x \in [0,1]$ and $\delta > 0$ we can find an $m \geq 1$ and an $[a,b] \in \mathcal{F}(m)$ such that $x \in [a-(b-a), b+(b-a)]$ and $\delta > b-a$.*

(iv) *F is nowhere differentiable.*

Proof. (i) Easy. Observe that $\|f_{n+1} - f_n\|_\infty \leq 3^{-1}\|f_n - f_{n-1}\|_\infty$.

(ii) Follows from (i) or from the relation between f_{n+m} and f_m obtained in the paragraph above.

(iii) Observe that either $x \in \bigcup_{[a,b]\in\mathcal{F}(n)}[a,b]$, $x \notin \bigcup_{[a,b]\in\mathcal{F}(n+1)}[a,b]$ for some n and apply Lemma 1.1(iii) and (iii)$'$, or $x \in \bigcup_{[a,b]\in\mathcal{F}(n)}[a,b]$ for all n and the result is trivial.

(iv) Let x, δ and $[a,b]$ be as in (iii). Then

$$F((a+b)/2) - F(x) = F(a) + (b-a) - F(x) = F(b) + (b-a) - F(x).$$

Since $|x-a|$, $|x-b|$, $|x-(a+b)/2| \leq 2(b-a)$ and $\delta > 0$ is arbitrary this is not consistent with differentiability at x. ∎

The following two related "devil's ramps" are much less well known than the staircase.

THEOREM 1.3 (Pompeiu). *There is a continuous function $g : [0,1] \to \mathbf{R}$ which is differentiable everywhere with bounded derivative, which is strictly increasing and yet has zero derivative on a dense subset.*

Proof. For the proof which is "quite elementary and understandable" see [9], p. 83. ■

By stretching the points of zero derivative into intervals or by modifying the technique of Katznelson and Stromberg set out in [9], pp. 80 to 83, or [5] we obtain the following function.

THEOREM 1.4 (Zahorski). *There is a continuous function* $g : [0,1] \to \mathbf{R}$ *together with a collection* $\mathcal{G}$ *of disjoint closed subintervals of* $[0,1]$ *having the following properties.*

(i) $g(0) = 0, \ g(1) = 1, \ g'(0) = g'(1) = 0.$

(ii) g *is increasing.*

(iii) $\bigcup_{[a,b]\in\mathcal{G}}[a,b]$ *is dense in* $[0,1]$.

(iv) g *is differentiable and there exists a* K *such that* $0 \le g'(x) \le K$ *for all* $x \in [0,1]$.

(v) g *is constant on each* $[a,b] \in \mathcal{G}$.

Let us see what happens if we define f, $\mathcal{F}$, f_0 and $\mathcal{F}(n)$ as before but this time set

$$f_n(x) = f(x) + \sum_{[a,b]\in\mathcal{F}} (b-a)^\alpha \chi_{[a,b]}(x) f_{n-1}((x-a)/(b-a)).$$

THEOREM 1.5. *Let* $\alpha > 1$. *Then*

(i) *The sequence* f_n *converges uniformly on* $[0,1]$ *to a continuous function* F *satisfying the relation*

$$F(x) = f(x) + \sum_{[a,b]\in\mathcal{F}} (b-a)^\alpha \chi_{[a,b]}(x) F((x-a)/(b-a))$$

for all $x \in [0,1]$.

(ii) $F(x) = F(a) + (b-a)^\alpha F((x-a)/(b-a))$ *for all* $x \in [a,b]$, $[a,b] \in \mathcal{F}(m)$.

(iii) *If* $x \in [0,1]$ *and* $\delta > 0$ *we can find an* $m \ge 1$ *and an* $[a,b] \in \mathcal{F}(m)$ *such that* $(x-\delta, x+\delta) \supseteq [a,b]$.

(iv) *If* $x \in [0,1]$ *and* $\delta > 0$ *then* F *is not monotonic in the interval* $(x-\delta, x+\delta) \cap [0,1]$.

(v) F *is differentiable with bounded derivative.*

Proof. Parts (i) to (iv) echo Theorem 1.2. To prove (v) observe that if θ is the largest length of any interval in $\mathcal{F}$ then $\theta < 1/2$ and

$$\sup_{x\in[0,1]} |f'_{n+1}(x) - f'_n(x)| \le \sup_{x\in[0,1]} |f'_n(x) - f'_{n-1}(x)|.$$

Thus f'_n converges uniformly to some bounded function h and standard manipulations using the mean value theorem (see e.g. last paragraph but one of p. 84 of [9]) show that F is differentiable with derivative h. ■

The existence of a nowhere monotone function with bounded derivative just proved is a famous theorem of Köthe. The elegant proof of Katznelson and Stromberg is given in [5]. A Baire category proof was given by Weil [10].

The remainder of this paper is devoted to embroidering the themes above. The proofs are left to the reader since one of my claims is that they are simple enough to be left to the reader. However, I should be glad to send on request the M.S. of my first draft in which the details are spelt out at great length. (*Caveat emptor*—the first draft has not been refereed, revised or checked.)

The topics dealt with in this paper have an enormous and, to all but the most persevering, hidden literature. I have relied on [2] and [9] for background and apologise in advance for any inadequacies in attribution. I hope that the general conception of this paper is original even if many of the specific results are not. (Even here it must be admitted that something along these general lines must be hidden in Besicovitch's original paper [1].)

2. Best staircases and ramps. From our point of view the best behaved version of the staircase of Lemma 1.1 that I can obtain is the following.

LEMMA 2.1. *Let $\psi : [0,\infty) \to [0,\infty)$ be a continuous increasing function with $t^{-1}\psi(t) \geq 2$ for all $0 \leq t \leq 1$ and $t^{-1}\psi(t) \to \infty$ as $t \to 0+$ and let $1 > \varepsilon > 0$. Then we can find a continuous function $g : [0,1] \to \mathbf{R}$ together with a collection $\mathcal{G}$ of disjoint subintervals of $[0,1]$ having the following properties.*

(i) $g(0) = 0,\ \ g(1) = 1$.

(ii) *g is increasing.*

(iii) *If $x \in [0,1)$ but $x \notin \bigcup_{[a,b]\in\mathcal{G}}[a,b]$ then given any $\delta > 0$ we can find an $[a,b] \in \mathcal{G}$ such that $(1+\delta)(b-a) \geq b - x \geq 0$ and $\delta \geq b - a$.*

(iii)′ *If $x \in (0,1]$ but $x \notin \bigcup_{[a,b]\in\mathcal{G}}[a,b]$ then given any $\delta > 0$ we can find an $[a,b] \in \mathcal{G}$ such that $(1+\delta)(b-a) \geq x - a \geq 0$ and $\delta \geq b - a$.*

(iv) *g is constant on each $[a,b] \in \mathcal{G}$.*

(v) *$|g(x) - g(y)| \leq \psi(|x-y|)$ for all $x, y \in [0,1]$.*

(vi) *If $[a,b] \in \mathcal{G}$ then $b - a < \varepsilon$.*

(vii) *If $[a,b] \in \mathcal{G}$ then $a \neq 0,\ \ b \neq 1$.*

Proof. This is tedious but routine. I suggest that my readers do not stop to prove it now but return to it after reading the rest of the paper if they need to cross every t. ■

Conditions (vi) and (vii) are merely technical conveniences and the improvement in (iii) and (iii)′ is not very important. The basic improvement is the Lipschitz smoothness condition on g given in (v). We can make g Lipschitz 1 at the price of weakening conditions (iii) and (iii)$'$.

LEMMA 2.1*. *As for Lemma* 2.1 *but with condition* (iii) *replaced by*

(iii)* *If* $x \in [0,1)$ *but* $x \notin \bigcup_{[a,b]\in\mathcal{G}}[a,b]$ *then given any* $\delta > 0$ *we can find an* $[a,b] \in \mathcal{G}$ *such that* $\psi(b-a) \geq b - x \geq 0$ *and* $\delta \geq b - a$,

condition (iii)$'$ *modified similarly to* (iii)$'^*$ *and condition* (v) *replaced by*

(v)* $|g(x) - g(y)| \leq (1+\varepsilon)|x - y|$.

Proof. Much as for Lemma 2.1. ∎

It is, fairly obviously, impossible to combine Lemmas 2.1 and 2.1*.

LEMMA 2.2. *Let* $K > 0$. *Suppose* $g : [0,1] \to \mathbf{R}$ *is a continuous function and* $\mathcal{G}$ *a collection of disjoint closed subintervals of* $[0,1]$ *with the following properties.*

(i) g *is constant on each* $[a,b] \in \mathcal{G}$.
(ii) $|g(x) - g(y)| \leq K|x-y|$ *for all* $x, y \in [0,1]$.
(iii) *If* $x \in (0,1]$ *but* $x \notin \bigcup_{[a,b]\in\mathcal{G}}[a,b]$ *then given any* $\delta > 0$ *we can find an* $[a,b] \in \mathcal{G}$ *such that* $K(b-a) \geq x - a \geq 0$ *and* $\delta \geq b - a$.

Then g *is constant.*

Proof. Use (iii) and the Lebesgue density theorem ([9], Theorem 21.29) to show that, in the sense of Lebesgue measure, almost all $x \in [0,1]$ lie in $\bigcup_{[a,b]\in\mathcal{G}}[a,b]$. Now use (i) and (ii). ∎

However, we can improve Lemma 2.1* by converting the staircase into a ramp.

LEMMA 2.1**. *As for Lemma* 2.1* *but with condition* (v)* *replaced by*

(v)** g *is everywhere differentiable with* $0 \leq g'(x) \leq 1 + \varepsilon$ *and* $g'(0) = g'(1) = 0$.

(It is perhaps worth remarking that the mean value theorem now shows that $\max(1, x+\varepsilon) \geq g(x) \geq \min(0, x - \varepsilon)$ for all $x \in [0,1]$.)

Proof. This follows from a general result of Bruckner (Theorem 6.8, Chapter II of [2]). However, it is easy (though the detail, as always, is tedious) to construct such a function by hand either by stretching Pompeiu's construction ([9], p. 83) or by modifying the technique of Katznelson and Stromberg ([9], p. 80). Again I suggest that readers defer further consideration until at least Section 3 below has been read. ∎

3. A selection of roller coasters. It is clear that the function of Lemma 2.1 can be modified to obey the extra condition

(viii) There exists an $\eta > 0$ such that $g(t) = 0$ for $0 \le t \le \eta$

and that the function and interval collection of Lemma 2.1** can be made to satisfy

(viii)** There exists an $\eta > 0$ such that $g(t) = 0$ for $0 \le t \le \eta$ and $[0, \eta] \setminus \bigcup_{[a,b] \in \mathcal{G}} [a, b]$ has positive Lebesgue measure.

Now choose $\psi(t) = -t \log t$ for t close to 0 (so that $t^{-\alpha}\psi(t) \to 0$ as $t \to 0+$ for all $\alpha < 1$). We now form our hump function $f : [0, 1] \to \mathbf{R}$ by $f(t) = f(1-t) = g(2t)$ for all $0 \le t \le 1/2$ and the associated $\mathcal{F}$ just as before.

THEOREM 3.1**. *Suppose we form f from the g of Lemma 2.1** modified to obey (viii)** and set $f_0 = 0$,*

$$f_n(x) = f(x) + \sum_{[a,b] \in \mathcal{F}} (b-a)^\alpha \chi_{[a,b]}(x) f_{n-1}((x-a)/(b-a))$$

for all $x \in [0, 1]$, $n \ge 1$.

(i) *If $\alpha < 0$ and r is the real number such that $\sum_{[a,b] \in \mathcal{F}} (b-a)^{\alpha r + 1} = 1$ then f_n converges pointwise almost everywhere to a measurable function F with $F \in L^p$ for all $1 \le p < r$. However, if I is any subinterval of $[0, 1]$ then $\chi_I F \notin L^r$.*

(ii) *If $\alpha = 0$ then f_n converges pointwise almost everywhere to a measurable function F with $F \in L^p$ for all $1 \le p < \infty$. However, if I is any subinterval of $[0, 1]$ then $\chi_I F \notin L^\infty$.*

(iii) *If $0 < \alpha \le 1$ then f_n tends uniformly to a continuous function F with*

$$\sup_x \sup_{h \neq 0} |F(x+h) - F(x)| \, |h|^{-\alpha} < \infty.$$

However, there exists a $\lambda > 0$ (depending on α) such that each subinterval of $[0, 1]$ contains an uncountable set of points y with

$$\limsup_{h \to 0} |F(y+h) - F(y)| \, |h|^{-\alpha} \ge \lambda > \liminf_{h \to 0} |F(y+h) - F(y)| \, |h|^{-\alpha} = 0.$$

If $0 < \alpha < \beta$ then

$$\limsup_{h \to 0} |F(x+h) - F(x)| \, |h|^{-\beta} = \infty \quad \text{and} \quad \liminf_{h \to 0} |F(x+h) - F(x)| \, |h|^{-\beta} = 0$$

for all $x \in [0, 1]$.

(iv) *If $\alpha > 1$ then f_n tends uniformly to a nowhere monotone differentiable function F with bounded derivative on any subinterval of $[0, 1]$. Every subinterval of $[0, 1]$ contains a set of a positive Lebesgue measure on which F' is zero.*

THEOREM 3.1. *Suppose we start with the g of Lemma 2.1 modified to obey* (viii) *and set* $f_0 = 0$,

$$f_n(x) = f(x) + \sum_{[a,b]\in\mathcal{F}} (b-a)^\alpha \chi_{[a,b]}(x) f_{n-1}((x-a)/(b-a))$$

for all $x \in [0,1]$, $n \geq 1$.

(i) *If* $\alpha \leq 0$ *then* $f_n(x) \to \infty$ *almost everywhere.*

(ii) *If* $0 < \alpha < 1$ *then* f_n *tends uniformly to a continuous function* F *with*

$$\sup_x \sup_{h\neq 0} |F(x+h) - F(x)|\,|h|^{-\alpha} < \infty.$$

However, there exists a $\lambda > 0$ *(depending on* α*) such that*

$$\limsup_{h\to 0} |F(x+h) - F(x)|\,|h|^{-\alpha} \geq \lambda > \liminf_{h\to 0} |F(x+h) - F(x)|\,|h|^{-\alpha} = 0$$

for all $x \in [0,1]$.

(iii) *If* $\alpha = 1$ *then* f_n *tends uniformly to a continuous function* F *which is nowhere differentiable on* $[0,1]$.

(iv) *If* $\alpha > 1$ *then* f_n *tends uniformly to a continuous function* F *of bounded variation which has derivative zero almost everywhere but is not monotone on any subinterval of* $[0,1]$.

Proofs. The claim of this paper is that these proofs may be left to the reader. ∎

(Note in connection with (iv) that if $\sup_x \sup_{h\neq 0} |G(x+h)-G(x)|\,|h|^{-1} < \infty$ then G is of bounded variation in every interval and so differentiable almost everywhere. A function exhibiting the kind of behaviour shown in Theorem 3.1 (ii) has been constructed by Kahane using the methods of [4].)

There is no reason why we should confine ourselves to humps. For example we could return to staircases and ramps by taking $\mathcal{F}$, $\mathcal{F}(m)$ as before, defining $f : \mathbf{R} \to \mathbf{R}$ by setting $f(t) = 0$ for $t \leq 0$, $f(t) = g(2t)$ for $0 \leq t \leq 1/2$, $f(t) = 1 + g(2t-1)$ for $1/2 \leq t \leq 1$ and $f(t) = 2$ for $t \geq 1$, and taking $f_0(t) = 0$,

$$f_n(t) = f(t) + \sum_{[a,b]\in\mathcal{F}} (b-a)^\alpha f_{n-1}((t-a)/(b-a))$$

for all $t \in \mathbf{R}$. I leave the investigation to the reader observing that it should produce among other things the following examples.

LEMMA 3.2. (i) *There exists a strictly increasing differentiable function* $F : [0,1] \to \mathbf{R}$ *with bounded derivative such that each subinterval of* $[0,1]$ *contains a set of positive measure on which* F' *is zero.*

(ii) *Let* $\psi : [0,\infty) \to [0,\infty)$ *be a continuous increasing function with* $t^{-1}\psi(t) \to \infty$ *as* $t \to 0+$ *and* $\psi(t) \geq 2t$ *for all* $t \geq 0$. *Then there exists a strictly increasing function* $F : [0,1] \to \mathbf{R}$ *which is almost everywhere differentiable with derivative zero and satisfies*

$$\sup_x \sup_{h\neq 0} |F(x+h) - F(x)|\, \psi(h)^{-1} < \infty.$$

Another kind of modification is illustrated in the next lemma.

LEMMA 3.3. *Let* $0 < \theta < 1$. *For each* $n \geq 1$ *let* $g_n : \mathbf{R} \to \mathbf{R}$ *be a continuous function of bounded variation and let* $\mathcal{G}(n)$ *be a family of disjoint closed subintervals of* $[0,1]$ *with the following properties.*

(i) $|g_n(t)| \leq 1$ *for all* $t \in [0,1]$, $n \geq 1$, $g_n(0) = g_n(1)$.
(ii) g_n *is constant on each* $[a,b] \in \mathcal{G}(n)$.
(iii) *If* $[a,b] \in \mathcal{G}(n)$ *then* $b - a < \theta$.
(iv) $\sum_{[a,b]\in\mathcal{G}(n)}(b-a) = 1$.

Set $\mathcal{F}(0) = \{[0,1]\}$, $f_0 = 0$,

$$\mathcal{F}(n) = \{\, [a + a'(b-a), a + b'(b-a)] : [a,b] \in \mathcal{F}(n-1),\ [a',b'] \in \mathcal{G}(n)\,\},$$
$$f_n(x) = f_{n-1}(x) + \sum_{[a,b]\in\mathcal{F}(n-1)} (b-a)\chi_{[a,b]}(x) g_n((x-a)/(b-a))$$

for all $x \in [0,1]$, $n \geq 1$.

Then f_n *converges uniformly to a continuous function* F *which obeys the following two dichotomies.*

(A) *Either* F *is of bounded variation on* $[0,1]$ *or* F *is not of bounded variation on any subinterval of* $[0,1]$.

(B) *Either* F *is differentiable with derivative zero almost everywhere or* F *has zero derivative almost nowhere on* $[0,1]$.

Proof. Observe that we can find a continuous function $F_n : [0,1] \to \mathbf{R}$ such that

$$F(x) = f_n(x) + \sum_{[a,b]\in\mathcal{F}(n-1)} (b-a)\chi_{[a,b]}(x) F_n((x-a)/(b-a))$$

and so $F(x) = F(a) + (b-a)F_n((x-a)/(b-a))$ for all $x \in [a,b]$, $[a,b] \in \mathcal{F}(n-1)$. Now use standard arguments. ∎

At first glance Lemma 3.3 seems to yield four possible outcomes. However, simple modifications of a theorem of Fubini ([9], Theorem 4.14) show that if F is of bounded variation then F has derivative zero almost everywhere. Theorem 3.1 (iii) and (iv) shows that two of the remaining possibilities occur. I leave it to the reader to show that the remaining possibility can occur.

LEMMA 3.4. *Let $\phi : [0,\infty) \to [0,\infty)$ be an increasing continuous function such that $t^{-1}\phi(t) \to 0$ as $t \to 0+$. Then we can choose g_n and $\mathcal{G}(n)$ in Lemma 3.3 in such a way that*

(i) *F has derivative zero almost everywhere.*
(ii) *F is not of bounded variation on any subinterval of $[0,1]$.*
(iii) *There exists a K such that if $0 \le x_0 < x_1 < \ldots < x_n \le 1$ then $\sum_{j=1}^{n} \phi(|F(x_j) - F(x_{j-1})|) \le K$.*

4. Besicovitch's nowhere differentiable function. If $F : \mathbf{R} \to \mathbf{R}$ is a function then for each $x \in \mathbf{R}$ we can define the upper right derivative

$$D^+F(x) = \limsup_{h \to 0+} \frac{F(x+h) - F(x)}{h}$$

and lower right derivative

$$D_+F(x) = \liminf_{h \to 0+} \frac{F(x+h) - F(x)}{h}.$$

The corresponding left derivatives D^-F, D_-F are obtained by replacing "$h \to 0+$" by "$h \to 0-$". In 1925 Besicovitch proved the following result.

THEOREM 4.1. *There exists a continuous function $F : \mathbf{R} \to \mathbf{R}$ such that $D^+F(x) > D_+F(x)$ and $D^-F(x) > D_-F(x)$ for all x.*

Such functions are rare (in the sense that their restrictions to $[a,b]$ form a set of the first category in $C([a,b])$ under the uniform norm, see [2], Ch. XIII). Besicovitch's proof had the reputation for being difficult and was reworked by E. D. Pepper [7], A. P. Morse [6] and R. L. Jeffery [3] (Section 7.3). In this section we give yet another reworking based on [3] obtaining the stronger result of Morse [6].

THEOREM 4.2. *There exists a continuous function $G : \mathbf{R} \to \mathbf{R}$ such that $D^+G(x) > D_+G(x)$, $D^-G(x) > D_-G(x)$ and in addition $\max(D^+G(x), -D^+G(x)) = \max(D^-G(x), -D^-G(x)) = \infty$ for all $x \in \mathbf{R}$.*

In some sense this result is best possible. A simple argument based on supporting hyperplanes (i.e. lines of support) shows that we cannot hope, for example, to have $D^+G(x) \ge 0 \ge D_+G(x)$, $D^-G(x) \ge 0 \ge D_-G(x)$ for all x.

Our first task is to find a suitable staircase function.

LEMMA 4.3. *Let $1 > \alpha > 0$ and $\gamma > 1$ be such that $\alpha\gamma < 1$. Then we can find $1/2 > \lambda > 0$ together with a continuous function $g : [0,1] \to \mathbf{R}$ and a collection $\mathcal{G}$ of disjoint closed subintervals of $[0,1]$ having the following properties.*

(i) $g(0) = 0$, $g(1) = 1$.

(ii) *g is increasing.*

(iii) *If $x \in [0,1)$ but $x \notin \bigcup_{[a,b]\in\mathcal{G}}[a,b]$ then given any $\delta > 0$ we can find an $[a,b] \in \mathcal{G}$ such that $(b-a)^{1/\gamma} \geq b - x \geq 0$ and $\delta \geq b - a$.*

(iii)′ *If $x \in (0,1]$ but $x \notin \bigcup_{[a,b]\in\mathcal{G}}[a,b]$ then given any $\delta > 0$ we can find an $[a,b] \in \mathcal{G}$ such that $(b-a)^{1/\gamma} \geq x - a \geq 0$ and $\delta \geq b - a$.*

(iv) *g is constant on each $[a,b] \in \mathcal{G}$.*

(v) *For each $1 > c > 0$ we can find a $K(c)$ such that $|g(x) - g(y)| \leq K(c)|x-y|$ for all $x, y \in [c,1]$.*

(vi) *If $[a,b] \in \mathcal{G}$ then $a \neq 0,\ b \neq 1$.*

(vii) *$g(x) \geq \lambda x$ for all $x \in [0,1]$.*

(viii) *$g(b) - 2^{-\alpha}(b-a)^\alpha \geq \lambda b$ for all $[a,b] \in \mathcal{G}$.*

(ix) *$x^{-1}g(x) \to \infty$ as $x \to 0+$.*

(x) *We can find a sequence of intervals $[a_n, b_n] \in \mathcal{G}$ with $g(b_n) - 2^{-\alpha}(b_n - a_n)^\alpha = \lambda b_n$ for all $n \geq 1$ and with $b_n \to 0$ as $n \to \infty$.*

(The new conditions (vii) to (x) are intended to give us firm control over the behaviour of $g(x)$ as $x \to 0+$. It will be clear from the proof that we could replace (v) by

(v)* g is differentiable with bounded derivative on each $[c,1]$ with $c > 0$,

but we do not need to.)

Proof. Take $\lambda = 8^{-1}$ and set $a_0 = 1,\ a_n = 2^{-2n-1},\ b_n = 2^{-2n}$ for each $n \geq 1$. Define $g(0) = 0,\ g(1) = 1$ and $g(x) = \lambda b_n + 2^{-\alpha}(b_n - a_n)^\alpha$ for $x \in [a_n, b_n]$ and $n \geq 1$ (thus automatically satisfying (x)).

By rescaling Lemma 2.1* or Lemma 2.1** we know that given any $\varepsilon(n) > 0$ we can define g on $[b_n, a_{n-1}]$ together with a family $\mathcal{G}(n)$ of disjoint closed subintervals of $[b_n, a_{n-1}]$ in such a way that

$(\mathrm{i})_n$ g is continuous on $[b_n, a_{n-1}]$.

$(\mathrm{ii})_n$ g is increasing on $[b_n, a_{n-1}]$.

$(\mathrm{iii})_n$ If $x \in (b_n, a_{n-1}]$ but $x \notin \bigcup_{[a,b]\in\mathcal{G}(n)}[a,b]$ then given any $\delta > 0$ we can find an $[a,b] \in \mathcal{G}$ such that $(b-a)^{1/\gamma} \geq x - a \geq 0$ and $\delta > b - a$.

$(\mathrm{iii})'_n$ The standard mirror condition of $(\mathrm{iii})_n$ holds.

$(\mathrm{iv})_n$ There exists a $K(n)$ such that $|g(x) - g(y)| \leq K(n)|x-y|$ for all $x, y \in [b_n, a_{n-1}]$.

$(\mathrm{v})_n$ If $[a,b] \in \mathcal{G}(n)$ then $a \neq b_n,\ b \neq b_{n-1}$.

$(\mathrm{vi})_n$ If $[a,b] \in \mathcal{G}(n)$ then $|b-a| \leq \varepsilon(n)$.

We have now defined g as a continuous increasing function on $[0,1]$. We take $\mathcal{G} = \{\, [a_n, b_n] : n \geq 1 \,\} \cup \bigcup_{n=1}^\infty \mathcal{G}(n)$ and observe that conditions (i) to (vi) are automatically satisfied. Next we observe that if $x \in [a_n, a_{n-1}]$ then

$$x^{-1}g(x) \geq a_{n-1}^{-1}g(a_n) \geq 2^{-\alpha}a_{n-1}^{-1}(b_n - a_n)^\alpha \geq 2^{2n-1}2^{-(2n+1)\alpha}.$$

Thus (ix) holds and $g(x) \geq 2\lambda x$ for all $x \in [0,1]$. Provided we take $\varepsilon(n)$ small enough conditions (vii) and (viii) follow at once. ■

We now form our roller coaster.

THEOREM 4.4. *Let g and $\mathcal{G}$ be as in Lemma 3.3. Set $f(t) = g(2t)$ for $0 \leq t \leq 1/2$, $f(t) = f(1-t)$ for $1/2 \leq t \leq 1$ and write*

$$\mathcal{F} = \bigcup \{\, [a/2, b/2], [1-b/2, 1-a/2] : [a,b] \in \mathcal{G} \,\}.$$

Let $f_0 = 0$, $\mathcal{F}(0) = [0,1]$,

$$f_n(x) = f(x) - \sum_{[a,b]\in\mathcal{F}} (b-a)^\alpha \chi_{[a,b]}(x) f_{n-1}((x-a)/(b-a)) \qquad \textit{for all } x \in [0,1],$$

$$\mathcal{F}(n) = \{\, [a' + a(b'-a'), a' + b(b'-a')] : [a,b] \in \mathcal{F},\ [a',b'] \in \mathcal{F}(n-1) \,\}.$$

Write $E(n) = \bigcup \{\, [a,b) : [a,b] \in \mathcal{F}(n) \,\}$.

(i) *f_n tends uniformly to a continuous function F with $F(0) = F(1) = 0$, $F(1/2) = 1$, $F(1-x) = F(x)$ satisfying*

$$F(x) = f(x) - \sum_{[a,b]\in\mathcal{F}} (b-a)^\alpha F((x-a)/(b-a)) \qquad \textit{for all } x \in [0,1].$$

(ii) $1 \geq f(x) \geq F(x) \geq f_2(x) \geq 2\lambda x$ *for* $x \in [0, 1/2]$.

(iii) *If $x \in E(1)$ then we can find y, z with $x < y < z \leq 1$ such that*

$$(F(y) - F(x))/(y-x) \geq 0, \qquad (F(z) - F(x))/(z-x) \leq 0,$$
$$\max((F(y) - F(x))/(y-x), |(F(z) - F(x))/(z-x)|) \geq \lambda.$$

(iv) *If $x \notin E(1)$ and $x \neq 0,1$ then $D_+F(x) = -\infty$, $D^+F(x) > -\infty$.*

(v) $D^+F(0) = \infty$, $D_+F(0) \leq 4\lambda$.

(vi) *If $x \in \bigcap_{m=1}^\infty E(m)$ then $D^+F(x) \geq 0 \geq D_+F(x)$ and* $\max(D^+F(x), |D_+F(x)|) = \infty$.

(vii) *If $x \in E(m-1) \setminus E(m)$ then $D^+F(x) > D_+F(x)$ and* $\max(D^+F(x), |D_+F(x)|) = \infty$.

(viii) *If $x \in [0,1)$ then $D^+F(x) > D_+F(x)$ and* $\max(D^+F(x), |D_+F(x)|) = \infty$.

(ix) *If $x \in (0,1]$ then $D^-F(x) > D_-F(x)$ and* $\max(D^-F(x), |D_-F(x)|) = \infty$.

(x) *If G is the periodic function of period 1 given by $G(x) = F(x)$ for $x \in [0,1]$ then G satisfies the conditions of Theorem 4.2.*

Proof. (The reader is advised to sketch f_1, f_2 and F.)

(i) Standard.

(ii) Conditions (vii) and (viii) of Lemma 4.3 are designed so that $f_2(x) \geq 2\lambda x$ for $x \in [0, 1/2]$. Thus $f_2(x) \geq 0$ for all $x \in [0, 1]$ and since

$$F(x) = f_2(x) + \sum_{[a,b] \in \mathcal{F}(2)} (b-a)^{\alpha} \chi_{[a,b]}(x) F((x-a)/(b-a))$$

we must have $F(x) \geq 0$ for $x \in [0, 1]$ and the result follows.

(iii) If $0 \leq x < 1/2$ take $y = 1/2$, $z = 1$. If $x \geq 1/2$ then $x \in [a, b)$ for some $[a, b] \in \mathcal{F}$. Take $y = b$, $z = 1$ and use condition (vii) of Lemma 4.3.

(iv) Since $x \in (0, 1)$ we know from Lemma 4.3(v) that we can find $\eta > 0$ and $K > 0$ such that whenever $|y - x| < \eta$ we have $|f(x) - f(y)| \leq K|x - y|$. By condition $(\mathrm{iii})_n$ or $(\mathrm{iii})'_n$ we can find $[a_n, b_n] \in \mathcal{F}$ such that $(b_n - a_n)^{1/\gamma} \geq x - a_n \geq 0$ and $b_n - a_n \to 0$. Set $c_n = (a_n + b_n)/2$ and observe that

$$F(c_n) = f(c_n) - (b_n - a_n)^{\alpha} F(1/2) = f(c_n) - (b_n - a_n)^{\alpha},$$
$$F(x) = f(x), \qquad F(b_n) = f(b_n).$$

Thus for large n

$$(F(b_n) - F(x))/(b_n - x) = (f(b_n) - f(x))/(b_n - x) \leq K$$

whilst

$$(F(c_n) - F(x))/(c_n - x) \leq K - (b_n - a_n)^{\alpha}/(b_n - x) \leq K - (b_n - x)^{\alpha\gamma - 1} \to -\infty$$

as $n \to \infty$. The result follows.

(v) Conditions (ix) and (x) of Lemma 4.3 were chosen specifically to make this true. By (x) we can find a sequence $[a_n, b_n] \in \mathcal{G}$ with $f(b_n) - (b_n - a_n)^{\alpha} = 2\lambda b_n$. Setting $c_n = (a_n + b_n)/2$ we have $F(b_n) = f(b_n)$, $F(c_n) = f(c_n) - (b_n - a_n)^{\alpha} = 2\lambda b_n$ so $(F(b_n) - F(0))/b_n = f(b_n)/b_n \to \infty$ as $n \to \infty$ by Lemma 4.3 (ix), and $(F(c_n) - F(0))/c_n = 2\lambda b_n/c_n \leq 4\lambda$. The result follows.

(vi) Rescale (iii).

(vii) Rescale (iv) and (v).

(viii) This is just (vi) and (vii) combined.

(ix) Symmetry.

(x) Obvious. ∎

Remark. Throughout this proof we have to keep in mind that "$K > L$" does not imply "$K_\infty > L_\infty$" unless $K \geq 0 \geq L$.

5. Elaborations. The results of Sections 4 and 5 when put together suggest the following three results.

THEOREM 5.1. *Let $\psi : \mathbf{R} \to \mathbf{R}$ be a continuous function with $\psi(t) = \psi(-t)$ for all t, $\psi(0) = 0$, $\psi(t)$ increasing and $t^{-1}\psi(t)$ decreasing as t runs from 0 to ∞, $t^{-1}\psi(t) \to \infty$ as $t \to 0+$. Then we can find a continuous function*

$F:\mathbf{R}\to\mathbf{R}$ satisfying

$$\sup_x \sup_{h\neq 0} |F(x+h)-F(x)|\psi(h)^{-1} < \infty$$

such that $F(x)=F(-x)$, $D^+F(x) > D_+F(x)$ and $\max(D^+F(x), |D_+F(x)|) = \infty$ for each $x\in\mathbf{R}$.

THEOREM 5.2. *Under the hypotheses of Theorem 5.1 we can find a continuous function $F:\mathbf{R}\to\mathbf{R}$ satisfying*

$$\sup_x \sup_{h\neq 0} |F(x+h)-F(x)|\psi(h)^{-1} < \infty$$

such that $F(x)=F(-x)$ and

$$\limsup_{h\to 0+}(F(x+h)-F(x))\psi(h)^{-1} \geq 0 \geq \liminf_{h\to 0+}(F(x+h)-F(x))\psi(h)^{-1},$$

$$\limsup_{h\to 0+}(F(x+h)-F(x))\psi(h)^{-1} - \liminf_{h\to 0+}(F(x+h)-F(x))\psi(h)^{-1} \geq 1,$$

for each $x\in\mathbf{R}$.

THEOREM 5.3. *Let ψ and ϕ both satisfy the conditions set out in the first sentence of Theorem 5.1. Suppose further that $\phi(t)^{-1}\psi(t)\to\infty$ as $t\to 0$. Then we can find a continuous function $F:\mathbf{R}\to\mathbf{R}$ satisfying*

$$\sup_x \sup_{h\neq 0} |F(x+h)-F(x)|\psi(h)^{-1} < \infty$$

such that $F(x)=F(-x)$ and

$$\limsup_{h\to 0+}(F(x+h)-F(x))\phi(h)^{-1} \geq 0 \geq \liminf_{h\to 0+}(F(x+h)-F(x))\phi(h)^{-1},$$

$$\limsup_{h\to 0+}(F(x+h)-F(x))\phi(h)^{-1} - \liminf_{h\to 0+}(F(x+h)-F(x))\phi(h)^{-1} = \infty,$$

for each $x\in\mathbf{R}$.

The following three trivial remarks show that, at least in some directions, these results are best possible.

(1) At a maximum $\limsup_{h\to 0+}(F(x+h)-F(x))\psi(h)^{-1}\leq 0$.

(2) If $\sup_x\sup_{h\neq 0}|F(x+h)-F(x)|\psi(h)^{-1}<\infty$ for all ψ of the type specified in Theorem 5.1 then in fact $\sup_x\sup_{h\neq 0}|F(x+h)-F(x)||h|^{-1}<\infty$ so F is locally of bounded variation and so differentiable almost everywhere.

(3) If F is uniformly continuous (and so, in particular if F is continuous and periodic) then we can find a ψ of the type specified in Theorem 5.1 such that $\sup_x\sup_{h\neq 0}|F(x+h)-F(x)|\psi(h)^{-1}<\infty$.

If $\psi(t)=t^\beta$, $\phi(t)=t^\gamma$ then Theorems 5.1 to 5.3 can all be obtained in the manner of Section 4 by considering limits of iterations of the form

$$f_n(x) = f(x) - \sum_{[a,b]\in\mathcal{F}} (b-a)^\alpha \chi_{[a,b]}(x) f_{n-1}((x-a)/(b-a))$$

for suitable f. However, if we allow more general ψ and ϕ then we can no longer use similarity ideas and the proofs involve a vast amount of book keeping. They are thus hard in the sense of being intricate though not in the sense of involving new ideas. I suspect that few readers would have the inclination or the patience to read the proof and that all of those few could construct their own versions without much trouble. (If not I should be glad to communicate the details.)

If the reader wishes to develop the ideas of this paper further it is possible that one direction is indicated by the work of Rogers and Taylor described in [8], Chapter 3, §3.

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