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ON NON-ASSOCIATIVE GROUPOIDS

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In [3] G. Grätzer and J. Płonka have showed that for every algebra $\mathfrak{A}$, having a binary idempotent Abelian and non-associative operation, we have $\omega_{n+1}(\mathfrak{A}) \geqslant \omega_n(\mathfrak{A})+n-1$ for all n; $\omega_n(\mathfrak{A})$ denotes the number of all essentially n-ary operations in the algebra $\mathfrak{A}$. In this note we show that

$$\omega_{n+1}(\mathfrak{A}) \geqslant \omega_n(\mathfrak{A})+\frac{2}{3}\left(2^{n-1}-(-1)^{n-1}\right).$$

The example of the groupoid $(G; 2x+2y)$, where $(G; +)$ is the Abelian group of the exponent 3 shows that this estimation is best possible (cf. [4]).

THEOREM. *Let* $\mathfrak{A} = (A; F\cup\{\cdot\})$ *be an algebra, where* $\cdot$ *is a binary commutative idempotent and non-associative operation. Then, for every natural* n, *we have*

$$\omega_{n+1}(\mathfrak{A}) \geqslant \omega_n(\mathfrak{A})+\frac{2}{3}\left(2^{n-1}-(-1)^{n-1}\right).$$

Proof. First of all let us observe that: if f is an essentially n-ary operation in $\mathfrak{A}$, then $f(x_1x_2, x_3, \ldots, x_{n+1})$ is essentially $(n+1)$-ary; if $f(x_1x_2, x_3, \ldots, x_{n+1}) = g(x_1x_2, x_3, \ldots, x_{n+1})$, then $f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n)$; if f is not an algebraic operation in the algebra $(A; \cdot)$, then $f(x_1x_2, x_3, \ldots, x_{n+1})$ is neither. Therefore, the sequence a_n, being the number of all essentially n-ary operations in $\mathfrak{A}$ which are not algebraic operations in the algebra $(A; \cdot)$, is non-decreasing; consequently, it is enough to prove that the inequality

$$\omega_{n+1}(A; \cdot) \geqslant \omega_n(A; \cdot)+\frac{2}{3}\left(2^{n-1}-(-1)^{n-1}\right)$$

holds for every n.

Case A. No simple iteration (see [1]) of $\cdot$ admits a non-trivial permutation (*trivial* means identity permutation and the transposition (1, 2)). One can write $\omega_n(A; \cdot) = a_n+b_n$, where $a_n = n!/2$ and b_n is the number of all essentially operations in $(A; \cdot)$ which do not equal $s_n^{\gamma} = s_n(x_{i_1}, \ldots, x_{i_n})$, where $\gamma = (i_1, \ldots, i_n)$. We have

$$\omega_{n+1}(A;\ \cdot) - \omega_n(A;\ \cdot) = \frac{(n+1)!}{2} - \frac{n!}{2} + (b_{n+1} - b_n) = \frac{n!}{2} n + c_n.$$

We show that $c_n \geqslant 0$ for every n.

Indeed, let f be an essentially n-ary operation such that $f \neq s_n^\mu$ for each permutation μ of n letters. Then $f(x_1 x_2, x_3, \ldots, x_{n+1}) = g$ is essentially $(n+1)$-ary operation and $g \neq s_{n+1}\mu'$ for every permutation μ' on $n+1$ letters. In fact, we have already remarked that g is an essentially $(n+1)$-ary operation. Suppose that

$$f(x_1 x_2, x_3, \ldots, x_{n+1}) = s_{n+1}(x_{i_1}, x_{i_2}, \ldots, x_{i_{n+1}}).$$

Since the left-hand side admits a transposition $(1, 2)$, the right one does as well. If $\{1, 2\} = \{i_1, i_2\}$, then putting $x_1 = x_{i_1}$ and $x_2 = x_{i_2}$ we obtain a contradiction, because $f \neq s_n^\mu$. But in the case $\{1, 2\} \neq \{i_1, i_2\}$ we get also a contradiction, because s_{n+1} does admit a non-trivial permutation and this is not the case. Consequently, $c_n = b_{n+1} - b_n \geqslant 0$. Now easy induction yields

$$\frac{n!}{2} n \geqslant \frac{2}{3}\left(2^{n-1} - (-1)^{n-1}\right),$$

which proves the case A.

Case B. There exists an n such that the simple iteration s_n admits a non-trivial permutation. Without loss of generality we can assume that n is the smallest number which has this property. Then, in view of Theorem 1 of [1], the groupoid $(A;\ \cdot)$ has to satisfy the law

$$(*) \qquad s_n(x_1, x_2, \ldots, x_n) = s_n(x_n, x_2, \ldots, x_1).$$

By Theorem 3 of [1], the groupoid $(A;\ \cdot)$ is a sum of direct systems of some groupoids which satisfy $(*)$ and the identity

$$(**) \qquad s_{n-1}(x, y, y, \ldots, y) = x.$$

It follows from [2] that

$$\omega_n(A;\ \cdot) = \sum_{k=0}^{n} \binom{n}{k} \omega_k^*,$$

where ω_n^* is the number of essentially n-ary operations in non-trivial groupoids from the class satisfying $(*)$ and $(**)$. Hence

$$\omega_{n+1}(A;\ \cdot) - \omega_n(A;\ \cdot) = \sum_{k=0}^{n+1} \binom{n+1}{k} \omega_k^* - \sum_{k=0}^{n} \binom{n}{k} \omega_k^*$$

$$= \sum_{k=0}^{n} \left(\binom{n+1}{k} - \binom{n}{k}\right) \omega_k^* + \omega_{n+1}^* \geqslant \omega_{n+1}^*.$$

By Theorem 2 of [1] every groupoid which satisfies $(*)$ and $(**)$ is of the form $(G;\ dx+dy)$, where $(G;\ +)$ is an Abelian group of exponent $2d-1$, which divides $2^{n-2}-1$. Since the operation $\cdot$ is non-associative, we have $2d-1\geqslant 3$. Indeed, if this is not the case, then every groupoid of the sum of direct systems would be trivial and, consequently, $(A;\ \cdot)$ would be a semilattice, because if each algebra of direct systems satisfies an identity, then also the sum does. Thus $d\geqslant 2$.

Now, the number w_n of all essentially n-ary idempotent operations in an Abelian group of exponent a is given by the formula

$$w_n = \frac{a^n-(-1)^n}{a+1}$$

(this is an unpublished result of B. Wolk). Thus, we have

$$\omega_{n+1}(A;\ \cdot)-\omega_n(A;\ \cdot)\geqslant \omega_{n+1}^{*} = \frac{(2d-1)^{n+1}-(-1)^{n+1}}{2d} \quad \text{for } d\geqslant 2.$$

Using induction on n, one can check that, for every n and $d\geqslant 2$,

$$\frac{(2d-1)^{n+1}-(-1)^{n+1}}{2d}\geqslant \frac{2}{3}\left(2^{n-1}-(-1)^{n-1}\right)$$

and the theorem follows.

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