COLLOQUIUM MATHEMATICUM

VOL. LII 1987 FASC. 1

ON LOCALLY m-CONVEX ALGEBRAS OF TYPE ES

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All algebras in this note are assumed to be commutative with unit element e. A *locally m-convex algebra* is a topological algebra whose topology can be determined by a family of submultiplicative seminorms. A locally m-convex algebra F is called an *ES-algebra* (shortly $F \in ES$) if for every subalgebra E of F every continuous multiplicative linear functional on E can be extended to such a functional on F.

A characterization of B_0-algebras of type ES has been established by Żelazko [5]. The aim of this note is to study characterizations of locally m-convex algebras of type ES.

Let $\mathfrak{Q}$ be a class of locally m-convex algebras and let $Q \in \mathfrak{Q}$. The algebra Q is said to *have the extension property with respect to* $\mathfrak{Q}$ (shortly $Q \in EP(\mathfrak{Q})$) if for every closed subalgebra E of F, $E \in \mathfrak{Q}$, every continuous homomorphism from E into Q can be extended to a continuous homomorphism from F into Q.

In Section 1, using a method of Żelazko [5], we give characterizations of ES-algebras. In Section 2 we prove that $C[z_1, \ldots, z_n] \in ES$ if and only if $n = 1$, where $C[z_1, \ldots, z_n]$ denotes the locally m-convex algebra of complex polynomials in n variables equipped with the compact-open topology. Section 3 is devoted to prove that a semi-simple Fréchet–Montel ES-algebra Q has the extension property with respect to the class of all metric ES-algebras if and only if Q is isomorphic to C^m for some $m \leqslant \infty$.

1. Characterizations of ES-algebras. For each locally m-convex algebra F we denote by $S(F)$ the set of all continuous submultiplicative seminorms p on F satisfying $p(e) = 1$. Let $p \in S(F)$. Put $F(p) = F/p^{-1}(0)$. Then $F(p)$ is a normed algebra with the norm p. If $\pi(p)$ denotes the canonical projection of F onto $F(p)$, then the spectrum $\sigma(x)$ of an element $x \in F$ is given by

$$\sigma(x) = \bigcup \{\sigma_p(x) \colon \ p \in S(F)\},$$

where $\sigma_p(x)$ denotes the spectrum of $\pi(p)x$ in the completion $(F(p))^{\wedge}$ of $F(p)$.

A locally m-convex algebra F is called a *Q-algebra* if the set of all invertible elements $G(F)$ of it is open.

THEOREM 1.1. *Let F be a locally m-convex algebra satisfying one of the following two conditions:*

(Q) *$[F, \tau]\hat{}$ is a Q-algebra under a topology stronger than the original one,*

(C) *$[F, \tau]$ is sequentially complete under a topology stronger than the original one.*

Then the following conditions are equivalent:

(i) *$F \in ES$,*

(ii) *$F(p) \in ES$ for every $p \in S(F)$,*

(iii) *for every $x \in F$ and for every $p \in S(F)$ the spectrum $\sigma_p(x)$ is totally disconnected.*

Proof. (a) Assume first that F satisfies (Q).

(i)$\Rightarrow$(iii): For a contradiction, there exist an $x \in F$ and a $p \in S(F)$ such that $\sigma_p(x)$ contains a continuum K. Let $\alpha, \beta \in K$, $\alpha \neq \beta$ and let us write $y = (x - \alpha e)/(\beta - \alpha)$. Then $\sigma_p(y)$ contains a continuum K with $0, 1 \in \tilde{K}$. Let

$$P_n(\lambda) = 1 + (i\lambda)/1! + \ldots + (i\lambda)^n/n!.$$

Since $\sigma_p(P_n(y)) = P_n \sigma_p(y)$, it follows that $1, P_n(1) \in K_n$, where $K_n = P_n(\tilde{K})$. Take N such that $\arg P_N(1) \neq 0$ and $Z_N = P_N(y) \in G([F, \tau]\hat{})$. By the connectedness, it follows that K_N contains λ_1 with $\arg \lambda_1 = 2\pi/n$ for some positive integer n. Let E be a subalgebra of F generated by (e, Z_N^n). Each $u \in E$ is of the form $u = P(Z_N^n)$, where $P \in C[z]$, and we put

$$f(P(Z_N^n)) = P(0):$$

Obviously f is a multiplicative linear functional on E. Since $\sigma_p(Z_N^n)$ separates the complex plane between 0 and ∞, by the maximum principle we have

$$\begin{aligned} |f(P(Z_N^n))| &= |P(0)| \leqslant \max\{|P(\lambda)|: \lambda \in \sigma_p(Z_N^n)\} \\ &= \max\{|\lambda|: \lambda \in \sigma_p(P(Z_N^n))\} \leqslant p(P(Z_N^n)). \end{aligned}$$

So f can be extended by continuity onto the whole of $[F, \tau]\hat{}$. On the other hand, f cannot be extended to a multiplicative linear functional on $[F, \tau]\hat{}$, since $f(Z_N^n) = 0$ and $Z_N^n \in G([F, \tau]\hat{})$.

(iii) $\Rightarrow$(ii): Let E be a closed subalgebra of $F(p)$, where $p \in S(F)$ and $x \in E$. Since $\sigma_p(x)$ does not contain a continuum, we have $\sigma_E(x) = \sigma_p(x)$ [3], where $\sigma_E(x)$ denotes the spectrum of x in E. Thus the spectrum of every element in E is disconnected. Applying Lemma 2 in [5] we have $\Gamma(E) = \mathfrak{M}(E)$, where $\Gamma(E)$ (resp. $\mathfrak{M}(E)$) denotes the Šilov boundary (resp. the space of all non-zero multiplicative linear functionals on E equipped with the weak topology) of E. Since every multiplicative linear functional on E belonging to $\Gamma(E)$ can be extended to such a functional on F, we get the implication (iii)$\Rightarrow$(ii).

(ii) $\Rightarrow$ (i) is trivial.

(b) Now assume that F satisfies condition (C). Then in notations of (a) we replace y by $\frac{\pi}{2}\frac{x-e}{\beta-\alpha}$ and Z_N by e^{iy} and by an argument similar as in (a) we get

$$(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i).$$

Theorem is proved.

COROLLARY 1.2 ([5]). *A locally m-convex B_0-algebra is an ES-algebra if and only if spectrum of any of its elements contains no continuum.*

Proof. By category arguments, Corollary 1.2 is an immediate consequence of Theorem 1.1.

COROLLARY 1.3. *Let F be a locally m-convex ES-algebra satisfying either of conditions* (Q) *and* (C) *in Theorem* 1.1 *and let f be a continuous homomorphism of F onto E. Then $E \in ES$.*

Proof. To prove Corollary 1.3 it suffices to check that $E(q) \in ES$ for every $q \in S(E)$. Let $q \in S(E)$. By the continuity of f there exists $p \in S(F)$ such that $q(f(x)) \leqslant p(x)$ for every $x \in F$. Thus f induces naturally a continuous homomorphism $\tilde{f}$: $F(p) \to E(q)$ such that $\operatorname{Im} \tilde{f} = E(q)$. Since $\sigma_q(y) \subseteq \sigma_p(x)$ holds for every $x \in F(p)$, $\tilde{f}(x) = y$, by Theorem 1.1 it follows that $E(q) \in ES$.

COROLLARY 1.4. *Let F_α, $\alpha \in \Omega$ be locally m-convex algebras satisfying either of conditions* (Q) *and* (C) *in Theorem* 1.1. *Then $F = \prod \{F_\alpha \colon \alpha \in \Omega\} \in ES$.*

Proof. Let $p \in S(F)$. Take a finite system $(p\alpha_1, \ldots, p\alpha_n)$ of $p\alpha_j \in S(F_j)$ such that $F(p) = F\alpha_1(p_1) \oplus \ldots \oplus F\alpha_n(p_n)$. By Theorem 1.1 and since

$$\sigma_{F(p)}(x) = \bigcup_{j=1}^{n} \sigma_{F\alpha_j(p\alpha_j)} \quad \text{for every } x = (x_1, \ldots, x_n) \in \prod_{j=1}^{n} F\alpha_j(p\alpha_j),$$

it follows that $F(p) \in ES$. Hence $F \in ES$.

Remark 1.5. There exists a normed algebra $A \in ES$ such that $\hat{A} \notin ES$.

Let Λ be the Cantor set in C and let $\operatorname{Lip}^\alpha(\Lambda)$ denote the Banach algebra of all Lipschitz functions of order α on Λ. Then $\operatorname{Lip}^\alpha(\Lambda) \in ES$ ([6]) and hence, by 1.3, $\operatorname{Im} \theta \in ES$, where θ denotes the canonical homomorphism of $\operatorname{Lip}^\alpha(\Lambda)$ into $C(\Lambda)$. On the other hand, since each continuum in C is a continuous image of Λ and $\overline{\operatorname{Im} \theta} = C(\Lambda)$ then by [6], or (iii) of Theorem 1.1 we have $C(\Lambda) \notin ES$.

2. An example of ES-algebras. Let $C[z_1, \ldots, z_n]$ be the locally m-convex algebra of complex polynomials in n variables equipped with the compact-open topology. We prove the following

PROPOSITION 2.1. *$C[z_1, \ldots, z_n] \in ES$ if and only if $n = 1$.*

Proof. Let A be a subalgebra of $C[z]$ and f a continuous multiplicative linear functional on A. Take a sequence $\{\varphi_k\}_{k=1}^{\infty}$ in A such that

$\varphi_1 \neq \text{const}$ and $\{\varphi_k\}$ is dense in A. For each k consider the map θ_k: $C \to C^k$ given by the formula

$$\theta_k(z) = (\varphi_1(z), \dots, \varphi_k(z)) \quad \text{for} \quad z \in C.$$

Put

$$\theta_\infty(z) = (\varphi_1(z), \dots, \varphi_k(z), \dots) \quad \text{for} \quad z \in C.$$

Since $\varphi_1 \neq \text{const}$, θ_k and θ_∞ are proper, i.e. $\theta_k^{-1}(K)$ and $\theta_\infty^{-1}(K)$ are compact for every compact set K in C^k and C^∞ respectively. Hence, by a theorem of Remmert ([1], Theorem 5, p. 162), $\operatorname{Im}\theta_k$ is an analytic set in C^k. Observe that $\operatorname{Im}\theta_\infty$ is closed in C^∞, where C^∞ is equipped with the product topology. Since every holomorphic function on an analytic set in C^k can be extended to a holomorphic function on C^k ([1], Theorem 18, p. 245) it follows that

$$\operatorname{Im}\theta_k = V(\operatorname{Ker}\hat{\theta}_k) \tag{2.1}$$

where $\hat{\theta}_k$ denotes the homomorphism from $\Theta(C^k)$, the space of holomorphic functions on C^k, equipped with the compact open topology, into $\Theta(C)$ induced by θ_k and

$$V(\operatorname{Ker}\hat{\theta}_k) = \{\omega \in C^k\colon \varphi(\omega) = 0 \text{ for } \varphi \in \operatorname{Ker}\hat{\theta}_k\}.$$

Since $f\hat{\theta}_k$ is a continuous multiplicative linear functional on $C[z_1, \dots, z_k]$, we have

$$f\hat{\theta}_k(\varphi) = \varphi(\omega^k) \quad \text{for} \quad \varphi \in C[z_1, \dots, z_k]$$

where $\omega^k = (f(\varphi_1), \dots, f(\varphi_k)) \in V(\operatorname{Ker}\hat{\theta}_k)$. Hence, by (2.1), we have $\omega^k = \theta_k z^k$ for some $z^k \in C$. Since

$$(f(\varphi_1), \dots, f(\varphi_k)) = \omega^k = \theta_k z^k = (\varphi_1(z^k), \dots, \varphi_k(z^k)),$$

we have

$$\begin{aligned}\theta_\infty(z^k) &= (\varphi_1(z^k), \dots, \varphi_k(z^k), \varphi_{k+1}(z^k), \dots)\\ &= (f(\varphi_1), \dots, f(\varphi_k), \varphi_{k+1}(z^k), \dots)\end{aligned}$$

and hence

$$\lim_k \theta_\infty(z^k) = \omega = (f(\varphi_1), \dots, f(\varphi_k), \dots).$$

On the other hand, since $\operatorname{Im}\theta_\infty$ is closed in C^∞, we have $\omega = \theta_\infty(z^0)$ for some $z^0 \in C$. Hence, setting $\tilde{f}(\varphi) = \varphi(z^0)$ for $\varphi \in C[z]$, we get the required extension of f.

Let $n \geqslant 2$. Consider the subalgebra A of $C[z_1, \dots, z_n]$ generated by z_1 and $z_1 z_2$. Since z_1 and $z_1 z_2$ are algebraically independent, the formula

$$f(\varphi) = \sum_{j=0}^{\infty} a_{0j},$$

where

$$\varphi(z_1, z_2) = \sum_{i \geq 1} a_{i0} z_1^i + \sum_{i,j \geq 1} a_{ij} z_1^i (z_1 z_2)^j + \sum_{j \geq 0} a_{0j} (z_1 z_2)^j \in A,$$

defines a multiplicative linear functional f on A. Since for every compact set K in C^n the seminorm

$$p(\varphi) = \sup_{(z_1, \ldots, z_n) \in K} \sum_{i_1, \ldots, i_n \geq 0} |a_{i_1 \ldots i_n} z_1^{i_1} \ldots z_n^{i_n}|$$

is continuous on $\Theta(C^n)$, it follows that f is continuous. On the other hand f cannot be extended to a continuous multiplicative linear functional $\tilde{f}$ on $C[z_1, \ldots, z_n]$, since otherwise

$$0 = \tilde{f}(z_1) \quad \text{and} \quad 1 = \tilde{f}(z_1 z_2) = \tilde{f}(z_1)\tilde{f}(z_2) = 0.$$

The proposition is proved.

Let V be an irreducible algebraic subset of C^n. By $C(V)$ we denote the locally m-convex algebra $C[z_1, \ldots, z_n]/I[V]$, where $I[V] = \{\varphi \in C[z_1, \ldots, z_n]: \sigma | V = 0\}$. It is known that if $k = \dim V$, then there exists a surjective map π: $V \to C^k$ such that $\hat{\pi}(C[z_1, \ldots, z_k]) \subset C[V]$ and $C[V]$ is integrally dependent on $\hat{\pi}(C[z_1, \ldots, z_k])$, i.e. every $u \in C[V]$ is a solution of some equation $X^n + a_1 X^{n-1} + \ldots + a_n$, where $a_1, \ldots, a_n \in \hat{\pi}(C[z_1, \ldots, z_k])$. This implies that $\hat{\pi}$ is an embedding of $C[z_1, \ldots, z_k]$ into $C[V]$. Thus by Proposition 2.1, $C[V] \in ES$ if $k \geq 2$.

Now consider the irreducible algebraic subset V of C^2 given by

$$V = \{(z_1, z_2) \in C^2: z_1 z_2 - 1 = 0\}.$$

By the maximum principle, the homomorphism π: $C[z_1] \to C[V]$, induced by the canonical projection π: $(z_1, z_2) \mapsto z_1$, is an embedding. Thus the multiplicative linear functional f on $\hat{\pi}(C[z_1])$ given by $f(\hat{\pi}\sigma) = \sigma(0)$ is continuous. The functional f cannot be extended to a multiplicative linear functional $\tilde{f}$ on $C[V]$, since $f(\hat{\pi} z_1) = 0$ and $\hat{\pi} z_1$ is invertible in $C[V]$.

Thus we get the following

COROLLARY 2.2. *Let V be an irreducible algebraic subset of C^n of dimension ≥ 2. Then $C[V] \notin ES$.*

Moreover, there exists an irreducible algebraic subset V of C^2 of dimension 1 such that $C[V] \notin ES$.

Remark 2.3. By Theorem 1.1, $C[z](p) \notin ES$ for some $p \in S(C[z])$. Hence by Proposition 2.1 it follows that the conditions (i) and (ii) in Theorem 1.1 are not equivalent without one of the hypotheses (Q) and (C).

3. Locally *m*-convex algebras having the *EP*(*MES*). By *MES* we denote the class of all metric *ES*-algebras. In this section we prove the following

THEOREM 3.1. *Let Q be a semi-simple Fréchet–Montel ES-algebra. Then*

Q has the $EP(MES)$ if and only if Q is isomorphic to the algebra C^m for some $m \leqslant \infty$.

We need the following

LEMMA 3.2 ([2], Lemma 1.4). *Let F be a Fréchet–Montel space. If there exists a continuous linear map from $\prod_{j=1}^{\infty} B_j$ onto F, where B_j are Banach spaces, then F is isomorphic to C^m for some $m \leqslant \infty$.*

Proof of Theorem 3.1. The fact that if $Q \cong C^m$, then Q has the $EP(MES)$ follows from the relations

$$C^n \in ES \quad \text{for every } n \geqslant 1.$$

Now assume that Q is a Fréchet–Montel algebra having the $EP(MES)$. We prove that $\bar{Q} = Q/\operatorname{Rad} Q$ is isomorphic to the algebra C^m for some $m \leqslant \infty$.

(a) Take an increasing sequence $\{p_n\}$ in $S(Q)$ determining the topology of Q. Theorem 1.1 implies that $Q_n = Q(p_n) \in ES$ for every $n \geqslant 1$ and hence, by Corollary 1.4, $\tilde{Q} = \prod_{n=1}^{\infty} Q_n \in ES$. Consider the canonical embedding of Q into $\tilde{Q}$

$$\theta x = \{\pi_n x\}_{n=1}^{\infty} \quad \text{for} \quad x \in Q$$

where $\pi_n = \pi(p_n)$. By hypothesis, there exists a continuous homomorphism $P\colon \tilde{Q} \to Q$ such that $P\theta = \mathrm{id}$. Thus by Lemma 3.2 we get

$$\dim Q(p) < \infty \quad \text{for every } p \in S(Q). \tag{3.1}$$

(b) Let us consider the canonical homomorphism $\bar{\theta}\colon \bar{Q} \to E$, where $E = \prod_{n=1}^{\infty} C(\mathfrak{M}(Q_n))$, given by the formula

$$\bar{\theta}(x + \operatorname{Rad} Q) = \{\beta_n \pi_n x\} \quad \text{for} \quad x + \operatorname{Rad} Q \in \bar{Q},$$

where $\beta_n\colon Q_n \to C(\mathfrak{M}(Q_n))$ are canonical homomorphisms. Note that β_n is surjective for every $n \geqslant 1$ and

$$\begin{cases} \operatorname{Rad} \prod_{n=1}^{\infty} Q_n = \prod_{n=1}^{\infty} \operatorname{Rad} Q_n, \\ P(\operatorname{Rad} \prod_{n=1}^{\infty} Q_n) \subset \operatorname{Rad} Q. \end{cases} \tag{3.2}$$

By (3.2) and by the openness of the homomorphism $\beta = (\beta_n)\colon \tilde{Q} \to E$ it follows that the formula

$$\bar{P}z = Pu + \operatorname{Rad} Q \quad \text{for} \quad z \in E,$$

where $u \in \tilde{Q}$, $\beta u = z$, defines a continuous homomorphism $\bar{P}\colon E \to \bar{Q}$ such that $\bar{P}\bar{\theta} = \mathrm{id}$. Thus $\bar{Q}$ can be considered as a closed subalgebra of E.

(c) Let $\{\tilde{p}_n\}$ be an increasing sequence in $S(E)$ determining the topology of E such that $\mathfrak{M}(E(\tilde{p}_n))$ separates points of $E(\tilde{p}_n)$ for every $n \geqslant 1$. Then $\mathfrak{M}(\bar{Q}(\tilde{q}_n))$, where $\tilde{q}_n = \tilde{p}_n | \bar{Q}$, separates points of $\bar{Q}(\tilde{q}_n)$ for every $n \geqslant 1$. Then by (3.1) we have

$$\bar{Q} = \varprojlim \bar{Q}(\tilde{q}_n) \cong \varprojlim C\big(\mathfrak{M}(\bar{Q}(\tilde{q}_n))\big) \cong C^m$$

for some $m \leqslant \infty$.

The theorem is proved.

Acknowledgement. I wish to express my gratitude to Professor W. Żelazko for his guidance, remarks and constant encouragements during the preparation of this paper.

REFERENCES

[1] R. Gunning and H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, N. J., 1965.

[2] Nguyen Van Khue, *On the cohomology groups of sheaves $\mathcal{S} \in L$*, Studia Mathematica 72 (1982), p. 183–197.

[3] C. E. Rickart, *General theory of Banach algebras*, Princeton 1960.

[4] W. Żelazko, *Concerning non-commutative Banach algebras of type ES*, Colloquium Mathematicum 20 (1969), p. 121–126.

[5] — *On m-convex B_0-algebras of type ES*, ibidem 20 (1969), p. 300–304.

[6] — *Concerning extension of multiplicative linear functionals in Banach algebras*, Studia Mathematica 31 (1968), p. 495–499.

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Reçu par la Rédaction le 15. 07. 1981