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ON TOPOLOGICAL FIELDS

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0. Introduction. The aim of this paper is to give a review of the theory of topological fields and a bibliography of the relevant papers. (All fields considered will be commutative.)

By a *topological field* we mean a field provided with a field topology, i.e. a topology in which the field operations $(x, y) \mapsto x \pm y$, $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous. Moreover, in the case where the field topology is non-discrete and non-trivial, we shall say that it is a *proper field topology*.

The first modern approach to topological fields is due to Kürschak [95] who introduced axioms for normed fields giving thus a unified theory for the real and complex numbers as well as for Hensel's [55] p-adic numbers. The next step has been made by several authors, who described the structure of normed fields (see [1]-[4], [54], [133]-[135], [149] and [160]). Van Dantzig [22] found all locally compact fields under some superfluous conditions (he assumed the second axiom of countability) and, independently, Pontriagin [138] described all locally compact and connected fields. The next important step in this direction was done in a short but very important paper of Šafarevič [157] who characterized fields with a topology induced by a norm. He has introduced also the important notion of a locally bounded field. Independently, this result was obtained by Kaplansky [75] in a more general form. Moreover, Zelinsky [182] gave a characterization of field topologies induced by a Krull valuation. The first example of a topological field in which the topology cannot be normed was given by Zelinsky [185].

Further investigators considered locally bounded fields (see [19], [40], [76], [84], [122], [125], [129]-[131], [170] and [171]). In 1952, Kowalsky [83] gave an elegant proof of the theorem of van Dantzig-Pontriagin.

In the last years, the attention was concentrated mainly on the locally unbounded fields and their properties (see [20], [38], [51], [56], [72], [84], [122] and [125]).

A complete description of all locally bounded field topologies was found by Nakano [129] who showed that they are induced by generalized valuations introduced by him.

We should mention also some papers with examples of connected fields of non-zero characteristic (see [14], [123] and [125]).

1. Normed fields. A field K is said to be *normed* if there exists a function $|\ |: K \to \boldsymbol{R}$, called a *norm* (or *real valuation*), such that, for all $x, y \in K$,

N_1. $|x| \geqslant 0$, $|x| = 0$ if and only if $x = 0$;

N_2. $|xy| = |x|\,|y|$;

N_3. $|x+y| \leqslant |x|+|y|$.

A norm is called *non-Archimedean* if it satisfies

N_3'. $|x+y| \leqslant \max\{|x|, |y|\}$.

The system of balls

$$U_\varepsilon = \{a \in K: |a| < \varepsilon\}, \qquad \varepsilon \in \boldsymbol{R},\ \varepsilon > 0,$$

defines a field topology in K.

Now let us consider a more general situation. Let K be any topological field provided with a topology $\mathcal{T}$. We write it, shortly, as $(K, \mathcal{T})$. If A and B are any two subsets of K, then AB denotes the set of all products ab, where $a \in A$ and $b \in B$. If $A \subset K$, then we say that A is *bounded* if, for every neighbourhood U of zero, there exists another one V such that $AV \subset U$. It is easy to see that this definition is equivalent to the following: for every neighbourhood U of zero, there exists $x \in K$ with $xA \subset U$. Let us remark that every compact set is bounded. Indeed, let A be compact. The continuity of multiplication implies that, for every $x \in A$, there exist neighbourhoods U_x and V_x of x and of the zero-element, respectively, with $U_x V_x \subset U$. Since A is compact, there exists a finite set $x_1, \ldots, x_n$ with $U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_n} \supset A$. Taking

$$V = V_{x_1} \cap V_{x_2} \cap \ldots \cap V_{x_n},$$

we have $AV \subset U$.

Now we list some properties of bounded sets:

(1) a subset of a bounded set is bounded;

(2) the closure of a bounded set is bounded;

(3) if A, B are bounded, then $A+B$, $A \cup B$ and $A \cdot B$, where

$$A \cdot B = \left\{\sum_{i=1}^{n} a_i b_i:\ a_i \in A,\ b_i \in B,\ n = 1, 2, 3, \ldots\right\},$$

are bounded;

(4) every convergent sequence (net) is bounded;

(5) if $x_a \to 0$ and (y_a) is bounded, then $x_a y_a \to 0$.

A topological field $(K, \mathcal{T})$ provided with a proper topology is called *locally bounded* if one of the following (equivalent) conditions is satisfied:

(B_1) there exists a bounded and open subset A of K;

(B_2) there exists a base of neighbourhoods of zero in K consisting of bounded open sets;

(B_3) for every bounded neighbourhood U of zero, the sets aU, $a \in K^\times = K \setminus \{0\}$, form a base of neighbourhoods of zero in K;

(B_4) the sets aR, $a \in K^\times$, form a base of neighbourhoods of zero in K, where $R \subset K$ is a set such that $0, 1 \in R$, $RR \subset R$, there exists an $a \in R^\times$ with $a(R+R) \subset R$, and $K = R(R^\times)^{-1}$ (see [27] and [29]).

Let now K be a non-Archimedean normed field. The set $V = \{x \in K\colon |x| \leqslant 1\}$ is called the *valuation ring*, and $P = \{x \in K\colon |x| < 1\}$ — the *valuation ideal*. It is easy to see that P is the only maximal ideal of V. Thus $\bar{K} = V/P$ is a field which is called the *residue-class field* of our norm. By the *value group* we will mean the multiplicative subgroup $|K^\times| = \{|x| \in \boldsymbol{R}\colon x \in K^\times\}$ of the positive reals $\boldsymbol{R}_+$. A norm is called *discrete* if its value group is a discrete subgroup of $\boldsymbol{R}_+$.

It is natural to ask the question, which field topologies can be induced by a real norm. Before answering it, let us recall that a *topological nilpotent* in a topological field $(K, \mathcal{T})$ is any element $x \in K$ for which $x^n \to 0$ for $n \to \infty$. An element $a \in K$ is *neutral* if neither a nor a^{-1} is a topological nilpotent.

The following theorem was first proved by Šafarevič [157]:

THEOREM 1. *Let $(K, \mathcal{T})$ be a topological field. Then the following conditions are equivalent*:

(i) *$\mathcal{T}$ is induced by a norm*;

(ii) *the set T of all topological nilpotents of K is open and $(K \setminus T)^{-1}$ is bounded*;

(iii) *the set T is open and $T \cup N$ is bounded, where N denotes the set of all neutral elements of K.*

If K is a normed field, then we may construct its completion $\hat{K}$, since K is a metric space. It is easy to see that $\hat{K}$ is a topological field. Note, however, that if $(K, \mathcal{T})$ is an arbitrary topological field, then it may happen that its completion $\hat{K}$ in $\mathcal{T}$ is not a field (see [20], [38], [70], [84], [122] and [184]). It seems worth to notice that up to 1951 the only known complete fields were normed (see [185]).

Now let us consider a problem of extending a norm from a subfield K of L to L. If K is a complete normed field and L is its finite extension, then the norm can be extended from K to L in a unique way. Indeed, it may be shown that if $|x|_K$ and $|x|_L$ are norms in K and L, respectively, then the norm in L is given by the formula (cf. [15] and [52])

$$|a|_L = |N_{L/K}(a)|_K^{1/N}, \quad \text{where } N = [L:K].$$

The converse implication — if a complete normed field L is an algebraic extension of a complete normed field K, then the degrees of elements of L over K are bounded — was proved in [75] and [133]. In particular, if L is a separable extension of K, then $[L:K]$ is finite. From Theorem 1 it follows, in particular, that $|x+iy| = (x^2+y^2)^{1/2}$ is the only extension of absolute value from $\boldsymbol{R}$ to $\boldsymbol{C}$. A well-known theorem of Gelfand-Mazur says that the complex-number field $\boldsymbol{C}$ is the only proper normed field extension of the reals $\boldsymbol{R}$ (see [113] and [161]). The theorem is valid also in the case of a pseudo-norm, i.e. where instead of $|xy| = |x|\,|y|$ only the properties

$$|xy|_L \leqslant |x|_L |y|_L \quad \text{and} \quad |rx|_L = |r|\,|x|_L, \quad r \in \boldsymbol{R},\ x, y \in L,$$

are assumed. On the other hand, if L is a finite extension of a normed field K, not necessarily complete, then the norm of K can be extended to a norm of L in finitely many non-equivalent ways, where any two norms defining the same topology in L are said to be equivalent. More details will be given in Chapter 3.

Now the question arises whether it is possible to describe all non-equivalent norms of a given field K. In the case of the rational-number field $\boldsymbol{Q}$ the answer is well known: the theorem of Ostrowski [134] states that the p-adic norms and the ordinary absolute value exhaust all non-equivalent and non-trivial norms of $\boldsymbol{Q}$. In the case of the rational-function field $K = k(x)$ with coefficients in k, every norm of K, which is trivial on k, is equivalent to a norm of the form $|\ |_p$ or to a norm of the type $|\ |_\infty$, where (cf. [181])

$$\left|\frac{f}{g}\right|_\infty = e^{\deg g - \deg f}, \quad f, g \in k[x].$$

In the case of a finite extension K of the rational-number field $\boldsymbol{Q}$, all norms are described as follows. The set R_K of all algebraic integers of K is a Dedekind domain. In this case there is a one-to-one correspondence between the non-zero prime ideals $\mathfrak{p}$ of R_K and the non-equivalent non-Archimedean norms of K. Indeed, fix an arbitrary prime ideal $\mathfrak{p}$ of R_K. Then

$$(a) = aR_K = \mathfrak{p}^n \prod_{\mathfrak{q} \neq \mathfrak{p}} \mathfrak{q}^{n_\mathfrak{q}(a)}, \quad n_\mathfrak{q}(a) \in \boldsymbol{N},$$

where $\mathfrak{q}$ runs over all prime ideals of R_K, and $n_{\mathfrak{q}}(a) \neq 0$ only for a finite number of $\mathfrak{q}$'s.

Now, if $b \in K$, $b = c/d$, where $c, d \in R_K$, then the formulae

$$|a|_{\mathfrak{p}} = e^{-n} \text{ for } a \in R_K \quad \text{and} \quad |b|_{\mathfrak{p}} = |c|_{\mathfrak{p}}/|d|_{\mathfrak{p}}$$

define a non-Archimedean norm of K. If $[K:Q] = n$, then there are exactly n embeddings of K into C, say, r_1 real embeddings and $2r_2$ non-real ones, $n = r_1 + 2r_2$. Let us take one embedding from every pair of the complex conjugate and all real embeddings and denote them by $g_1, g_2, \ldots, g_r$ ($r = r_1 + r_2$). It is easy to see that every function $|a|_i = |g_i(a)|$, where $|a|$ is the absolute value in C, is a norm of K. It turns out that every norm of K is equivalent to some of $|a|_i$ or to some $\mathfrak{p}$-adic norm $|a|_{\mathfrak{p}}$ (see [135]). In other cases a description of all norms of a given field K seems rather hopeless. It depends deeply upon the algebraic structure of K.

Let now K be an arbitrary normed field complete in a norm $|\ |_1$, and let $|\ |_2$ be any non-equivalent norm of K. Can the field K be complete also in the norm $|\ |_2$? An answer is contained in the following theorem going back to Schmidt [156] (see also [26], [77] and [159]):

THEOREM 2. *Let K be an arbitrary normed field complete in two inequivalent norms. Then K is an algebraically closed field. Conversely, every algebraically closed field K is complete in $2^{|K|}$ inequivalent norms.*

For example, in the case of the complex-number field, for an arbitrary automorphism g of C, $g \in \mathrm{Aut}(C)$, the formula $|z|_g = |g(z)|$ gives a norm of C. Different discontinuous automorphisms (i.e., other than identity or conjugate) define inequivalent norms of C. Hence the cardinality of such norms is equal to $|\mathrm{Aut}(C)| = 2^{\mathfrak{c}}$.

The following approximation theorem was proved by Artin and Whaples (cf. [3] and [4]):

THEOREM 3. *If there are given any n non-trivial, non-equivalent norms $|\ |_s$ of K, an element a_s for each norm and an $\varepsilon > 0$, then there is an element a of K such that $|a - a_s|_s < \varepsilon$ for each $s = 1, 2, \ldots, n$.*

This theorem implies, in particular, that if $\mathscr{T}$ is the supremum of a finite family of topologies induced by inequivalent norms of K, then the completion $\hat{K}$ of K in $\mathscr{T}$ is the direct sum of the completions of K in these norms.

The approximation theorem implies also at once that inequivalent norms are multiplicatively independent:

If $|\ |_1, |\ |_2, \ldots, |\ |_n$ are non-trivial and inequivalent, the relation

$$|x|_1^{k_1} |x|_2^{k_2} \ldots |x|_n^{k_n} = 1$$

is true for all non-zero $x \in K$ if and only if all $k_i = 0$.

This corollary allows an axiomatization of the valuation theory. A set of all equivalent and non-trivial valuations of a field K is called a *prime divisor* of K and is denoted by $\mathfrak{p}$, $\mathfrak{q}$ etc.

Axiom 1. There is a set $\mathfrak{M}$ of prime divisors and a fixed set of valuations $|\ |_{\mathfrak{p}}$, one for each $\mathfrak{p}\in\mathfrak{M}$, such that, for every $a\in K$, $a\neq 0$, $|a|_{\mathfrak{p}} = 1$ for all but a finite number of $\mathfrak{p}\in\mathfrak{M}$ and

$$\prod_{\mathfrak{p}\in\mathfrak{M}} |a|_{\mathfrak{p}} = 1.$$

Axiom 2. The set $\mathfrak{M}$ of Axiom 1 contains at least one prime divisor $\mathfrak{q}$ of the two following types:

1. Discrete, with a finite residue-class field.

2. Archimedean, with a complete field $K_{\mathfrak{q}}$ (completion of K) which is either the real- or the complex-number field.

It can be shown (see [3]) that if K is a field satisfying Axioms 1 and 2, then it is an extension of a finite degree either of the rational-number field Q or of the field of rational functions $k = k_0(X)$.

2. Locally compact fields. We shall consider now some special classes of normed fields. We have already shown that every locally compact field is locally bounded. More is true, namely, every locally compact field is a complete normed field. This can be deduced either from Theorem 1 or using the existence and uniqueness of Haar measure in the additive group K^+ of a locally compact field K. In fact, if μ is the Haar measure in K^+ and $E\subset K^+$ is any open subset of K^+ with $0<\mu(E)<\infty$, then the function $\mu_x(E) = \mu(xE)$ is also a Haar measure in K^+, and so there exists a constant, say $|x|$, such that

$$\mu_x(E) = |x|\mu(E) \quad \text{or} \quad |x| = \mu(xE)/\mu(E).$$

It can be shown (cf. [167]; details in [37]) that, for some $a>0$, $|x|^a$ is a norm inducing the original locally compact topology in K.

In 1931, van Dantzig [22] introduced the general notion of a topological field. He proved that if a proper locally compact field satisfies the second axiom of countability and is connected, then it is isomorphic either to the real- or to the complex-number field, and if it is completely disconnected (note that any disconnected topological field must be completely disconnected (see [13])), then it is isomorphic to a finite extension of k, where k is a p-adic number field or a Laurent series field $k_0\langle X\rangle$ over a finite field k_0. Independently, Pontriagin [138] showed the validity of this theorem in the connected case, but without conditions of countability type. Jacobson [63] considered disconnected, non-commutative locally compact topological fields and showed that they all have to be division algebras of a finite degree over Q_p or $k_0\langle X\rangle$. Moreover, he proved that every disconnected locally compact division ring can be normed. A general proof of the classification theorem for locally compact fields was given by Kowalsky [83] in 1953. His proof is based on the existence of topological nilpotents in a locally compact, non-discrete (hence non-compact) field (see also [118] and [120]). A classification of locally compact fields is given by the following

THEOREM 4. *Let $(K, \mathcal{T})$ be any proper topological field. Then K is a finite extension of one of the following fields:*

(1) *the reals $\boldsymbol{R}$,*

(2) *a p-adic number field Q_p,*

(3) *a formal Laurent series field $k_0\langle X\rangle$ over a finite field k_0.*

The generalizations of Theorem 4 will be given below.

Now, let K be any normed field, complete in a non-Archimedean norm. Then, under suitable additional conditions, K is determined by its residue-class field $\bar{K}$. Let us recall that a field K is *perfect* if its characteristic is zero or, in the case of characteristic $p \neq 0$, if $K^p = K$. Hasse and Schmidt [54] have shown that if K is a perfect field, complete in a discrete and non-Archimedean norm, then K is uniquely determined by its residue-class field $\bar{K}$, except the case where it has characteristic zero, the characteristic of $\bar{K}$ equals $p \neq 0$ and K is ramified over Q (see also [104]).

Now we will consider a more general class of topological fields than the normed fields, namely, fields with Krull-Schilling valuations.

3. The Krull-Schilling valuations. Let Γ be any multiplicative ordered group with added zero 0 such that $0\cdot\gamma = \gamma\cdot 0 = 0$ and $0 \leqslant \gamma$ for every $\gamma\in\Gamma$. A function $v\colon K\to\Gamma\cup\{0\}$ is called a *valuation of the field K* if the following conditions are satisfied:

V_1. $v(x)\geqslant 0$, $v(x) = 0$ if and only if $x = 0$;

V_2. $v(xy) = v(x)v(y)$;

V_3. there exists a $\lambda\in\Gamma$ such that

$$v(x+y)\leqslant \lambda\max\{v(x), v(y)\} \quad \text{for all } x, y\in K.$$

(A valuation v is called *non-Archimedean* if $\lambda = 1$, where 1 is the unit element of Γ.)

This notion has been introduced by Krull in his fundamental paper [88]. A great number of results in this direction has been obtained by Schilling, the author of monograph [155]. In the last years interesting results were proved by Ribenboim [140]-[144].

A valuation v of a field defines a field topology in K: we take the sets of the form $U(\gamma) = \{x\in K\colon v(x)<\gamma\}$, $\gamma\in\Gamma$, as a base of the neighbourhoods of zero in K. Similarly as in the case of a non-Archimedean normed field, we define a valuation ring R_v of v and a residue-class field $\bar{K}$ of v. Let us start with examples (cf. [155]).

I. Let $K = k(X_1, X_2, \ldots, X_n)$ be a rational-function field of the algebraically independent indeterminates $X_1, X_2, \ldots, X_n$ with coefficients from a field k. At first, we shall consider a ring of all polynomials $R = k[X_1, X_2, \ldots, X_n]$ in the variables $X_1, X_2, \ldots, X_n$. We agree to assign to the monomial $X_1^{m_1} X_2^{m_2} \ldots X_n^{m_n}$ the value

$$v(X_1^{m_1} X_2^{m_2} \ldots X_n^{m_n}) = (e^{-m_1}, e^{-m_2}, \ldots, e^{-m_n}) \in \boldsymbol{R}_+^n$$
$$= \{(r_1, r_2, \ldots, r_n) \colon r_j > 0 \text{ for } j = 1, 2, \ldots, n, v_j \in \boldsymbol{R}\}.$$

Next, we define a multiplication in $\Gamma = \boldsymbol{R}_+^n$:

$$(r_1, r_2, \ldots, r_n)(s_1, s_2, \ldots, s_n) = (r_1 s_1, r_2 s_2, \ldots, r_n s_n).$$

We order n-tuples of positive reals lexicographically so that

$$(r_1, r_2, \ldots, r_n) > (s_1, s_2, \ldots, s_n)$$

if there exists a k, $0 \leqslant k \leqslant n$, with $r_t = s_t$ for $t = 1, 2, \ldots, k$, $r_{k+1} > s_{k+1}$. Each polynomial $f(X_1, X_2, \ldots, X_n)$ can be expressed as

$$f(X_1, \ldots, X_n) = X_1^{m_1} F_1(X_1), \quad \text{where } F_1(X_1) \in k[X_2, \ldots, X_n][X_1].$$

Let $f_0(X_2, \ldots, X_n) = F(0)$ be a non-zero polynomial in $X_2, \ldots, X_n$. In the same way we can write

$$f_0(X_2, \ldots, X_n) = X_2^{m_2} F_2(X_2),$$
$$\text{where } F_2(X_2) \in k[X_3, \ldots, X_n][X_2] \text{ and } F_2(0) \neq 0.$$

Similarly, we can obtain the numbers $m_3, \ldots, m_n$. Let us put

$$v(f) = (e^{-m_1}, e^{-m_2}, \ldots, e^{-m_n}).$$

Since every element of K is of the form f/g, where $f, g \in R$ and $g \neq 0$, we can extend this definition of v to K by putting $v(f/g) = v(f)v(g)^{-1}$ and $v(0) = 0$. It is easy to verify that v is a valuation in K and $\Gamma \cup \{0\}$ is its value group.

II. Let k be an arbitrary field and let Γ be an arbitrary linearly ordered group. Let us consider the set K of all formal sums

$$\sum_{\varrho < \gamma} b_{a(\varrho)} X^{a(\varrho)},$$

where $\varrho > \tau$ implies $a(\varrho) > a(\tau)$, $b_{a(\varrho)} \in k$, with $X^a X^\beta = X^{a+\beta}$ for all $a, \beta \in \Gamma$.

Next, we define addition and multiplication in K in the usual formal fashion. It can be shown that we obtain again a field. Finally, we put

$v(b) = X^{a(1)}$ for $b \in K^{\times}$, $v(0) = 0$. A simple calculation shows that v is a valuation of K and its value group is order-isomorphic to Γ.

This example shows that every linearly ordered abelian group can be the value group of a suitably chosen valuation.

Now a question arises: which topological fields have valuations preserving their topologies?

The answer for non-Archimedean valuations was given by Zelinsky [182]:

THEOREM 5. *Let $(K, \mathcal{T})$ be a proper topological field. Then the following conditions are equivalent:*

(i) *$\mathcal{T}$ is induced by a non-Archimedean valuation;*

(ii) *a certain neighbourhood of zero generates an additive group which is bounded and, for every subset A of K, disjoint with some neighbourhood of zero, A^{-1} is bounded;*

(iii) *there is an open and bounded subring R of K containing 1 with K as its field of fractions and such that, for every subset A of K, disjoint with a certain neighbourhood of zero, A^{-1} is bounded.*

A multiplicative linearly ordered group Γ is called *Archimedean* if, for every $\alpha, \beta \in \Gamma$ with $\alpha > 1$, there exists an $n \in \boldsymbol{N}$ such that $\alpha^n \geqslant \beta$. It turns out (see [155]) that every linearly ordered Archimedean group is order-isomorphic to a subgroup of the multiplicative group $\boldsymbol{R}_+$ of positive reals. Hence, any valuation taking values in an Archimedean ordered group can be identified with a norm (real valuation).

If Γ is any linearly ordered group, we define the *absolute value* $|\gamma|$ in Γ as γ if $\gamma \geqslant e$ (e is the unit element of Γ), γ^{-1} for $\gamma < e$, and $|0| = 0$. A subgroup Δ of Γ is called *isolated* if Δ contains with δ all elements $\gamma \in \Gamma$ for which $|\gamma| \leqslant |\delta|$. Since the set of all isolated subgroups Δ of Γ, $\Delta \neq \Gamma$, is linearly ordered by inclusion, we can define the *rank of Γ* as the order type of the set of all isolated subgroups Δ of Γ, $\Delta \neq \Gamma$. By the *rank of valuation v* of a field K with the value group $\Gamma = v(K^{\times})$ we will mean the rank of Γ, $K^{\times} = K \setminus \{0\}$. We will consider now the problem of extension of a given valuation from a field K to its finite extension. Let K be an arbitrary field and let L be its finite extension of degree $[L: K] = n$. Let v be any valuation of K and $v_1, v_2, \ldots, v_g$ its inequivalent prolongations to L. Denote by Γ the value group of v and by Γ_i the value groups of v_i, $i = 1, 2, \ldots, g$. Similarly, let k and k_i denote residue-class fields of v and v_i, respectively. Then g is finite, $[k_i: k] < \infty$ and Γ has a finite index in each Γ_i, $(\Gamma_i: \Gamma) < \infty$. If, moreover, $f_i = [k_i: k]$, i.e., if the inertial degree of v relatively to v_i is f_i, and $e_i = (\Gamma_i: \Gamma)$ is the ramification index of v, then

$$\sum_{i=1}^{g} e_i f_i \leqslant n.$$

Finally, if L is a Galois extension of K (i.e., L is a separable and normal extension of K), then we have

$$n = p^{\delta} \sum_{i=1}^{g} e_i f_i,$$

where $p \neq 0$ is the characteristic of K, $\delta \geqslant 0$, and

$$\sum_{i=1}^{g} e_i f_i = n$$

in the case of the zero characteristic (cf. [18]).

Most results on normed fields can be carried over to the fields with valuations. As an example, let us consider a generalization of Theorem 3 given by Jaffard [67]:

THEOREM 6. *Suppose that $v_1, v_2, \ldots, v_n$ are non-discrete and pairwise inequivalent valuations of a field K. If $a_1, a_2, \ldots, a_n \in K^{\times}$ and $b_1, b_2, \ldots, b_n \in K$ are arbitrary, then there exists an $x \in K$ such that $v_i(x-b_i) = v_i(a_i)$ for all $i = 1, 2, \ldots, n$.*

To finish this chapter we present some generalizations of Theorem 2. Let K be a field with valuation v. Let R_v be its valuation ring and P_v the only prime ideal of R_v, i.e., the ideal of the valuation v. The field K is called *henselian* (or *relatively complete*) if the Hensel lemma is true in K for v, namely, if, for every polynomial $f(X) \in R_v[X]$ such that

$$f(X) \equiv g_0(X) h_0(X) \ (\mathrm{mod} P_v), \qquad \big(g_0(X), h_0(X)\big) \equiv 1 \ (\mathrm{mod} P_v),$$

there exist $g(X), h(X) \in R_v[X]$ such that

$$f(X) = g(X) h(X)$$

with

$$g(X) \equiv g_0(X) \ (\mathrm{mod} P_v) \quad \text{and} \quad h(X) \equiv h_0(X) \ (\mathrm{mod} P_v).$$

Every field K, complete with respect to the rank one of valuation v, is henselian, but not conversely (see [147] and [155]). A field K is *henselian with respect to the valuation v* if and only if v has exactly one prolongation to every algebraic extension of K.

The following generalization of Schmidt's theorem, given in [26], holds:

THEOREM 7. *A field K is relatively complete with respect to two inequivalent valuations if and only if it is separably algebraically closed in its completion* (see also [77] and [159]).

It can be proved that if an algebraic closure $\tilde{K}$ of a field K is its finite extension, then K is henselian with respect to v if and only if its residue-class field $\bar{K}$ is a really closed field (see [26]).

4. Pseudo-valuations. Semi-valuations. A further generalization of the notion of a norm is the notion of a pseudo-valuation (or a pseudo-norm). The pseudo-valuations were introduced and examined in a series of Mahler's papers [109].

A *pseudo-valuation* (or a *pseudo-norm*) on a field K is defined as a real-valued function $p\colon K\to \boldsymbol{R}_+$ such that

P_1. $p(x)\geqslant 0$, $p(0) = 0$, $p\not\equiv 0$;

P_2. $p(xy)\leqslant p(x)p(y)$;

P_3. $p(x-y)\leqslant p(x)+p(y)$.

A pseudo-valuation p is *non-Archimedean* if it satisfies the ultrametric inequality

P'_3. $p(x-y)\leqslant \max\{p(x), p(y)\}$.

It is not difficult to see that $x\neq 0$ implies $p(x)\neq 0$. In fact, the set $I = \{x\in K\colon p(x) = 0\}$ is an ideal in K, and since $p\not\equiv 0$ on K, we have $I = (0)$; thus $p(x) = 0$ implies $x = 0$.

Similarly, as in the case of normed fields, two pseudo-valuations p_1 and p_2 are called *equivalent*, $p_1\sim p_2$, if they define the same topology on K. Let us call a topology $\mathscr{T}$ on K a *PV-topology* if it can be defined by a pseudo-valuation.

The question arises what are the necessary and sufficient conditions for the field topology $\mathscr{T}$ of K to be induced by a pseudo-valuation. The answer is given by the following theorem of Cohn [19]:

THEOREM 8. *Let $(K, \mathscr{T})$ be a proper topological field. Then the following conditions are equivalent*:

(i) *$\mathscr{T}$ is a PV-topology*;

(ii) *K contains a non-empty open bounded set and there exists a non-zero nilpotent element in K*;

(iii) *K has a non-empty bounded set consisting of nilpotents.*

Moreover, a topology $\mathscr{T}$ can be induced by a non-Archimedean pseudo-valuation if and only if K has a non-empty open bounded subset closed under addition and containing nilpotent elements only.

The proofs are based on the notion of a gauge set. A subset A of K will be called a *gauge set of K* if $-1\in A$, $AA\subset A$, there exists a $c\in K$ such that $A+A\subset cA$, $A\neq K$, and there exists a $d\in K^\times$ with

$$\bigcup_{n=1}^{\infty} d^{-n}A = K.$$

There is a one-to-one correspondence between the set of all gauge sets on K and all pseudo-valuations on K. A subset B of K is said to be *linear* if, given $x\in K^\times$, either $x\in B$ or $x^{-1}\in B$. It turns out that a non-trivial pseudo-valuation p on a field K is equivalent to a valuation if and only if it is associated with a linear gauge set (see [19]). If K is a non-discrete pseudo-normed field of characteristic zero and p is its pseudo-norm such that $p(x) = |x|$ for every $x \in Q$, then K is algebraically isomorphic to a subfield of the complex-number field C (see [7] and [8]). The last assertion remains valid if $p(x) = |x|$ holds for infinitely many natural numbers $x \in N$ or if $p(x) = |x|$ for all x with $|x| < \delta$, $\delta > 0$ (see [10]).

If K is a pseudo-valued field for which there exists a number $n_0 > 1$, $n_0 \in N$, with $p(n^2) = p(n)^2$ for all $n \geqslant n_0$, $p(n_0) > 1$, then there is a continuous embedding of K into C. A pseudo-valuation p on K is said to be *multiplication monotone* if $p(x) < p(x')$ and $p(y) < p(y')$ imply $p(xy) \leqslant p(x'y')$. It can be shown that a multiplication monotone pseudo-norm of a field K is equivalent to a norm (see [9]).

Successive generalizations of the norm of a field lead to concepts of semi-valuation and half-valuation. It was done, among others, by Fuchs [45] and Jaffard [66]-[68].

Let Γ be a partially ordered, multiplicatively written commutative group. By a *half-valuation* we will mean any mapping $v\colon K^{\times} \to \Gamma$ such that

V_1'. $v(ab) = v(a)v(b)$;

V_2'. $v(a) \geqslant v(c)$ and $v(b) \geqslant v(c)$ implies $v(a-b) \geqslant v(c)$;

V_3'. $v(K^{\times}) = \Gamma$.

Let us recall that a partially ordered commutative group is called a *lattice-ordered group* or *l-group* if, for every $\alpha, \beta \in \Gamma$, there exist

$$\sup\{\alpha, \beta\} = \alpha \vee \beta \quad \text{and} \quad \inf\{\alpha, \beta\} = \alpha \wedge \beta, \quad \text{where } \alpha \leqslant \beta,$$

if and only if $\alpha \vee \beta = \beta$ $(\alpha \wedge \beta = \alpha)$.

A *semi-valuation* s on a field K is a mapping $s\colon K^{\times} \to \Gamma$, where Γ is an l-group, such that

S_1. $s(ab) = s(a)s(b)$;

S_2. $s(a+b) \geqslant \inf\{s(a), s(b)\}$;

S_3. $s(K^{\times}) = \Gamma$.

It can be proved that every abelian l-group is a semi-valuation group of some field [66].

We call a set Ω of positive elements in Γ an *upper class of Γ* if, for every $\alpha \in \Omega$, $\alpha \leqslant \beta$ implies $\beta \in \Omega$ and if, moreover, Ω is a sublattice of Γ. We shall call an upper class Ω of Γ an *l-closed upper class* if it contains the greatest lower bound to every set of any two elements in Ω. By an *order of a semi-valuation s* we mean the number of maximal l-closed upper classes of its value group. We shall say that *a semi-valuation is of the first kind* if its order is finite. If s is of order one, then it is a valuation.

Yakabe [179] is the author of the following extension theorem:

THEOREM 9. *If k is a subfield of a field K, then any semi-valuation s of the first kind and of order n on k can be extended to a semi-valuation of the first kind of K and of the same order n. In particular, if K is a finite extension of k, then the number of distinct extensions of s in K, which are of order n, is finite and not greater than d^n, where d is the degree of separability of K over k.*

5. Topologies of type V. In paper [75] Kaplansky introduced a notion of rings of type V. Let $(K, \mathcal{T})$ be any topological field. We shall say that a *subset $A \subset K$ is bounded away from zero* if it is disjoint with some neighbourhood of zero. The *topology $\mathcal{T}$ is of type V* if, for every subset A of K, bounded away from zero, A^{-1} is bounded. It can be shown that the completion $\hat{K}$ of a topological field K of type V is again a field (and of type V). We are going to characterize the topologies of type V. Before doing it let us recall some definitions.

Let $(K, \mathcal{T}_K)$ be any topological field and let $\mathcal{T}$ be an arbitrary topology on K. The topology $\mathcal{T}$ on K is said to be *admissible with respect to $\mathcal{T}_K$* if K endowed with $\mathcal{T}$ is a topological vector space over K provided with $\mathcal{T}_K$; this means that the mapping from $\mathcal{T} \times \mathcal{T}$ to $\mathcal{T}$ defined by $(x, y) \mapsto x+y$ is continuous, and that the mapping from $\mathcal{T}_K \times \mathcal{T}$ to $\mathcal{T}$ defined by $(x, y) \mapsto xy$ is also continuous. This implies that $\mathcal{T} \leqslant \mathcal{T}_K$. A topological field K is said to be *minimal* (or *full*) if its topology $\mathcal{T}_K$ is a minimal element in the ordered set of all field topologies on K; that is, if there exists no field topology $\mathcal{T}$ on K such that $\mathcal{T} < \mathcal{T}_K$. The topological field $(K, \mathcal{T}_K)$ is called *strictly minimal* if there exists no topology $\mathcal{T}$ on K admissible with respect to $\mathcal{T}_K$ and such that $\mathcal{T} < \mathcal{T}_K$; this means that the only topology on K admissible with respect to $\mathcal{T}_K$ is $\mathcal{T}_K$.

It is not difficult to see that topologies induced by the valuations and topologies of type V are minimal. We shall say that the topological field $(K, \mathcal{T}_K)$ is an *absolutely simple field* if every continuous non-zero homomorphism of it is a topological isomorphism. Some results in papers [40], [84] and [126] can be put together as follows:

THEOREM 10. *If $(K, \mathcal{T})$ is a topological field, then the following conditions are equivalent*:

(i) *$\mathcal{T}$ is a minimal topology*;

(ii) *the field $(K, \mathcal{T})$ is absolutely simple*;

(iii) *the completion $\hat{K}$ of K in a topology $\mathcal{T}$ is a field.*

Every strictly minimal topological field is minimal [127]. It can be proved that every field provided with a topology induced by a non-trivial valuation is strictly minimal.

In [127] it is proved the following

THEOREM 11. *Let* $(K, \mathscr{T})$ *be a topological field. Then the following conditions are equivalent*:

(i) *every finite-dimensional vector space over* K *has only one admissible topology*;

(ii) *every automorphism of any finite-dimensional topological vector space over* K *is continuous*;

(iii) K *is a strictly minimal and complete field.*

In order to finish this chapter let us recall the following characterization of topologies of type V given by Fleischer (cf. [39] and [40]) and Dürbaum and Kowalsky (cf. [29]):

THEOREM 12. *Let* $(K, \mathscr{T})$ *be a proper topological field. Then the following conditions are equivalent*:

(i) $\mathscr{T}$ *is of type* V;

(ii) *for every subsets* A *and* B *of* K, *if* A *and* B *are bounded away from zero, then also* AB *is bounded away from zero*;

(iii) $\mathscr{T}$ *is a minimal locally bounded field topology*;

(iv) $\mathscr{T}$ *is induced by a Krull valuation in* K.

The topologies of type V have many interesting properties. For instance, if $(K, \mathscr{T})$ is a topological field of characteristic not equal 2, and if $P(X) = X^2 + aX + b \in K[X]$, where $ab \neq 0$, then $P(X)$ has a continuous branch of the inverse function if and only if $\mathscr{T}$ is of type V. Moreover, if $\mathscr{T}$ is not of type V, then there exist topological fields for which the square-root function is discontinuous at every non-zero point [80].

The notion of a topology of type V plays an important role in the theory of topological fields. For any topological field $(K, \mathscr{T})$, we write $G(K)$ for the group of all its continuous automorphisms. The locally compact fields of characteristic zero can be described in terms of locally bounded topologies [171]:

THEOREM 13. *Let* $(K, \mathscr{T})$ *be a proper topological field. Then the following conditions are equivalent*:

(i) K *is a locally bounded, complete field and, for every closed subfield* F *of* K, $G(F)$ *is finite*;

(ii) K *is a locally compact field of characteristic zero*;

(iii) K *is a finite extension of the reals* $\boldsymbol{R}$ *or of some* p*-adic number field* Q_p *with the usual locally compact topology.*

For further generalizations of this theorem (for all locally compact fields), see [172].

6. The generalized valuations. A characterization of the locally bounded field topologies. The final step in a description of all locally bounded field topologies was made by Nakano in 1960 (see [129]-[131]). He introduced the notion of a generalized valuation which generalizes all known types of valuations, semi-valuations, norms etc. All locally bounded field topologies can be described in terms of generalized valuations. At first, some definitions. By a *value group of a generalized valuation* or, simply, by a *v-group* we shall mean a set Σ such that

G_1. Σ is a multiplicative group and 1 is its unit element;

G_2. in Σ, there is defined an associative and commutative addition $(\alpha, \beta) \mapsto \alpha + \beta$;

G_3. $\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma$ and $(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$ for every $\alpha, \beta, \gamma \in \Sigma$.

Let K be a field and let Σ be a v-group. A function $\| \ \|: K \to 2^{\Sigma}$ assigning to every element $x \in K$ a subset $\|x\|$ of Σ is called a *generalized valuation* if the following conditions are satisfied for all $x, y \in K$:

GV_1. $\|x\| = \Sigma$ if and only if $x = 0$;

GV_2. $\|x\| + \|y\| \subset \|x+y\|$;

GV_3. $\|x\| \cdot \|y\| \subset \|xy\|$.

Every generalized valuation induces a field topology on K, since all sets

$$U_\sigma = \{x \in K: \ \sigma \in \|x\| + \Sigma\}, \qquad \sigma \in \Sigma,$$

form a base of neighbourhoods of zero in K.

The main result of Nakano (see [129]) is the following

THEOREM 14. *For every topological field $(K, \mathcal{T})$, the following conditions are equivalent:*

(i) *$\mathcal{T}$ is a locally bounded topology;*

(ii) *$\mathcal{T}$ is induced by a suitable generalized valuation.*

Now, let Σ_i, $i \in I$, be a family of v-groups and let Σ be the direct product of all Σ_i. If addition and multiplication are defined componentwise, Σ becomes a v-group. If Σ_i are l-groups for all but a finite number of indices, then we may consider the restricted direct product Σ^* consisting of all elements of Σ such that all but a finite number of components are identities of Σ_i's. Σ^* is also a v-group. It can be shown that every locally bounded field topology is induced by a generalized valuation which takes values in the restricted direct product of the v-groups $\Sigma_i = \boldsymbol{Z}$ and $\Sigma_0 = \boldsymbol{R}_+$ [130]. Moreover, if I is infinite, then every restricted direct product of Σ_i ($\Sigma_i = \boldsymbol{Z}$ or $\boldsymbol{R}_+$) is the value group of some generalized valuation. If I is finite, a generalized valuation takes values in $\boldsymbol{R}_+$, i.e. it induces the same topology as a suitable pseudo-norm.

7. Connected fields. Topological characterizations of $\boldsymbol{R}$ and $\boldsymbol{C}$. As stated in chapter 2, the real- and complex-number fields are the only locally compact connected fields [138]. Dieudonné [25] gave an example of a connected subfield of $\boldsymbol{C}$ different from $\boldsymbol{R}$ and $\boldsymbol{C}$, and Kapuano [78] has proved the existence of a one-dimensional subfield L of $\boldsymbol{C}$, $L \neq \boldsymbol{R}$. This furnishes a counter-example to a conjecture of Baer and Hasse [13] that $\boldsymbol{R}$ is the only one-dimensional subfield of $\boldsymbol{C}$. In connection with the theorem of Pontriagin it is worth noticing that the following theorem remains true (see [125] and [170]):

THEOREM 15. *Every locally bounded complete and connected field is topologically isomorphic either to* $\boldsymbol{R}$ *or to* $\boldsymbol{C}$.

A field topology is called *locally convex* if there exists a base of neighbourhoods of zero consisting of convex sets, i.e. a base such that, for every neighbourhood U of zero and for every n,

$$\underbrace{U+U+\ldots+U}_{n\text{ times}} = n\cdot U = \{nu:\ u\in U\}.$$

It can be shown that every locally convex complete and connected field is topologically isomorphic either to $\boldsymbol{R}$ or to $\boldsymbol{C}$ (see [112]). Mutylin [122] has given a generalization of the Mazur-Gelfand theorem: $\boldsymbol{R}$ and $\boldsymbol{C}$ are the only locally bounded extensions of $\boldsymbol{R}$.

The complex-number field can be characterized also in the following way (see [168] and [170]):

THEOREM 16. *Let* $(K, \mathscr{T})$ *be a proper topological field. Then the following conditions are equivalent*:

(i) K *is topologically isomorphic to the complex-number field* $\boldsymbol{C}$;

(ii) K *is a locally bounded complete and algebraically closed field with* $G(K)$ *torsion and non-trivial*;

(iii) K *is a locally bounded complete and algebraically closed field with* $G(K)$ *finite and non-trivial.*

Moreover, Knopfmacher and Sinclair [82] showed recently that $\boldsymbol{C}$ and the real closed subfields Λ of $\boldsymbol{C}$ with $\Lambda(i) = \boldsymbol{C}$ are the only normed fields having a finite number of non-isomorphic extensions. This theorem remains valid also for fields with valuations [173].

In a natural way, the problem arises whether there are any connected fields of non-zero characteristic. Such examples were given by Mutylin (see [123] and [125]), Waterman and Bergmann (see [14]). They have shown that every discrete field can be embedded in a connected field in a suitably chosen topology. Indeed, let k be a field of arbitrary characteristic. Let us consider the ring R of polynomials in uncountably many variables T_a, i.e., $R = k[\{T_a\}_{a\in(0,1)}]$, $T_0 = 0$, $T_1 = 1$, topologized by taking the sets U_ε, $\varepsilon\in\boldsymbol{R}$, $\varepsilon > 0$,

$$U_\varepsilon = \bigcup (T_{a_1}-T_{\beta_1}, \ldots, T_{a_n}-T_{\beta_n}),$$

where the sum is taken over all sequences

$$a_1, \ldots, a_n, \beta_1, \ldots, \beta_n \in (0,1) \quad \text{with} \sum_i |a_i - \beta_i| < \varepsilon$$

for a base of zero-neighbourhoods in R. (Here $(x_1, x_2, \ldots, x_n)$ denotes the ideal in R generated by $x_1, x_2, \ldots, x_n$.) Then the function

$$|f| = \inf_{f \in U_\varepsilon} \varepsilon$$

defines a metric $d(f, g) = |f-g|$ in R, invariant under translations. It can be shown that the field of fractions of R, say K, becomes a connected field in a topology defined by a metric extending the metric $d(f, g)$ of R (see [14]). Examples given by Mutylin are similar.

8. Topologies on the rational-number field Q. Generalizations. It seems to be a hopeless but very interesting question to describe all the field topologies of a given field K in terms of its algebraic structure.

At first, let us start with the rationals Q. A classical theorem of Ostrowski [134] states that the ordinary absolute value and the p-adic norms are the only norms of Q (up to equivalency). Similarly, it can be proved that every locally bounded minimal field topology of Q is induced by a norm [122]. Kowalsky [84] was the first who has given an example of a locally unbounded topology on Q. Before showing his example let us recall some definitions.

Let $\mathfrak{A}$ be a family of field topologies of K,

$$\mathfrak{A} = \{\mathscr{T}_i \colon i \in I\}.$$

Let J be an arbitrary subset of I. A topology $\mathscr{T}_J$ is called the *intersection topology of* $\mathscr{T}_j$, where $j \in J$,

$$\mathscr{T}_J = \bigcap_{j \in J} \mathscr{T}_j,$$

if $\mathscr{T}_J$ is induced on K by the sets of the form $\bigcap' U_j$, where U_j is a neighbourhood of zero in a topology $\mathscr{T}_j$ and $\bigcap'$ means that $U_j = K$ for all but a finite number of j's. It is not difficult to show that $(K, \mathscr{T}_J)$ is a topological field [38]. Let now $\mathfrak{B}$ be a family of all locally bounded field topologies of K. It turns out that, for $\mathscr{T}_j \in \mathfrak{B}$, where $j \in J$,

$$\mathscr{T}_J = \bigcap_{j \in J} \mathscr{T}_j \in \mathfrak{B},$$

if and only if J is finite [84]. This theorem assures the existence of locally unbounded field topologies of Q. Indeed, let $P = \{p_1, p_2, \ldots\}$ be the set of all primes, and let us take an arbitrary infinite subset J of N. Denoting by $\mathscr{T}_j$ a suitable p_j-adic topology on Q, one can see that $\mathscr{T}_J$ is not locally bounded. Mutylin [122] gave other examples of locally unbounded field topologies in Q such that $\hat{Q}$'s are fields, contrary to examples of Kowalsky, and, moreover, there is at least continuum of such topologies on Q. Let $p_1, p_2, \ldots$ be any sequence of prime numbers. Mutylin constructs a topology $\mathscr{T}$ on Q satisfying some special conditions and requiring that $p_n \to 0$ in the topology $\mathscr{T}$. It is sufficient to take the sets

$$U_n = \left\{ \sum_{r=n}^{\infty} \frac{p_r k_r}{s_r} \colon k_r \in \boldsymbol{Z},\, s_r \in \boldsymbol{N},\, \left| \frac{k_r}{s_r} \right| \leqslant a_{rn},\, |s_r| \leqslant 2^{3^{r-n}} \right\}$$

for a base of neighbourhoods of zero for $\mathscr{T}$. Here all but a finite number of k_r are equal zero and a_{rn}'s satisfy certain conditions. For similar results, see [79].

Now let us consider a more general situation. Let A be a principal ideal domain and $\boldsymbol{P}$ a representative system of the irreducible elements of A. Denote by K the field of fractions of A and let $\mathscr{T}_p$ be a p-adic topology on K. Then the following theorem holds [20]:

THEOREM 17. *If $(K, \mathscr{T})$ is a proper topological field, then the open A-submodules of K form a fundamental system of neighbourhoods of zero if and only if there exists a subset J of $\boldsymbol{P}$ such that $\mathscr{T} = \mathscr{T}_J$, i.e. if*

$$\mathscr{T} = \bigcap_{p \in J} \mathscr{T}_p.$$

In particular, $K = Q$ for $A = \boldsymbol{Z}$ and thus we can describe all field topologies $\mathscr{T}$ of Q for which the additive subgroups of Q are neighbourhoods of zero in $\mathscr{T}$. This theorem remains valid if we take a Dedekind integral domain A instead of a principal ideal domain, as shown by Jebli [70] and, independently, by myself in my doctoral dissertation in 1971. It would be interesting to describe other classes of integral domains having similar properties.

9. Locally unbounded field topologies. The first example of such a topology gave Zelinsky [185]. In his example the topology was minimal. Another example was provided by Gould [51]. He topologized the rational-function field $\boldsymbol{R}(x)$ of an indeterminate x over the reals in a suitable way, but his example can be extended to an arbitrary rational-function field $k(x)$, where k is a proper locally bounded topological field (cf. [36], II). At first, the sets

$$U(\varepsilon) = \left\{ f(x) \in \boldsymbol{R}[x] \colon f(x) = \sum_{n \geqslant 0} a_n x^n,\ \forall_n\ |a|_n < \varepsilon_n \right\},$$

where $\varepsilon = (\varepsilon_n)$ is any sequence of positive real numbers, $\varepsilon_n \to 0$, are taken as a base of the neighbourhoods of zero for some ring topology of $\boldsymbol{R}[x]$.

Then the sets of the form

$$V(\varepsilon, g) = \frac{gU(\varepsilon)}{1+gU(\varepsilon)}, \quad \text{where } g(x) \in \boldsymbol{R}[x],$$

define a locally unbounded field topology for $\boldsymbol{R}(x)$ (see [51]).

Many results on locally unbounded field topologies can be found in Mutylin [125]. We present here some of them. It was shown by several authors (see [56] and [174]) that there are complete and locally unbounded extensions of fields $\boldsymbol{R}$, $\boldsymbol{C}$, of a discrete field and of locally unbounded fields. It still remains an open problem if there exists a complete locally unbounded extension of the p-adic number field Q_p. Our knowledge of locally unbounded fields is rather scarce. So far there is no classification of such fields even in the most simple case such as the rational-number field Q. Recently, interesting results obtained Heine (cf. [D], [E] and [F]) and Podewski (cf. [J] and [K]).

10. Final remarks. Open problems. In connection with the results mentioned in Chapter 7, the following questions arise:

1. Are $\boldsymbol{R}$ and $\boldsymbol{C}$ the only complete and connected fields ? (**P 880**)

2. Are $\boldsymbol{R}$ and $\boldsymbol{C}$ the only complete and connected fields with finite number of their continuous automorphisms? (**P 881**)

3. Is $\boldsymbol{C}$ the only non-discrete complete and algebraically closed field with finite number of continuous automorphisms? (**P 882**)

4. Does the approximation theorem of Artin-Whaples remain true if we take the minimal topologies instead of the locally bounded and minimal ones (i.e. valuations), that is, if, for any two minimal topologies $\mathcal{T}_1 \neq \mathcal{T}_2$ of a field K and for any neighbourhoods U_1 and U_2 of zero (respectively, in $\mathcal{T}_1$ and $\mathcal{T}_2$), there is

$$(U_1+a_1) \cap (U_2+a_2) \neq \emptyset$$

for any $a_1, a_2 \in K$. (**P 883**)

Obviously, this conjecture is not right in the case of non-minimal field topologies. Indeed, taking $\mathcal{T}_1 = \mathcal{T}_p \cap \mathcal{T}_q$ and $\mathcal{T}_2 = \mathcal{T}_p \cap \mathcal{T}_{q'}$, where p, q, q' are different primes in $\boldsymbol{Z}$, and $\mathcal{T}_p$ denotes the p-adic topology, one easily sees that $(U_1+a_1) \cap (U_2+a_2) = \emptyset$ for suitably chosen U_1, U_2, a_1, a_2.

5. Is Theorem 2 true for arbitrary minimal topologies, i.e., can a non-algebraically closed field be complete in two non-equivalent and non-discrete topologies? (**P 884**)

The positive answer to this question would imply that of 3 even under a weaker assumption that $G(K)$ is a non-discrete torsion group (cf. [170]).

6. Does there exist a complete and locally unbounded extension of the p-adic number field Q_p? (See [125].)

7. Are the locally compact fields of characteristic zero the only locally bounded and complete fields with a finite $G(K)$? (See [171].)

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