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# *OVER-RINGS OF AN (HNP)-RING*

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Let $R$ be an (hnp)-ring and $S$ be an over-ring of $R$. First of all the structure of idempotent ideals is studied. The results so proved are used to prove the followings. (i) If $R$ has enough invertible ideals, so does $S$ have. (ii) For any idempotent ideal $A$ of $S$, $A \cap R$ is an idempotent ideal of $R$.

In a recent paper Goodearl and Warfield ([5]) discussed in details the problem of extension of simple modules by simple modules over an (bnp)-ring, and they obtained a periodicity theorem ([5, Theorem 20]). In this note the results of Goodearl and Warfield are used to determine the structure of idempotent ideals of an (hnp)-ring $R$, and the results stated above.

**1. Preliminaries.** For basic properties of (hnp)-rings and modules over such rings, we refer to [2], [3]. Let $R$ be an (hnp)-ring and $Q$ be its classical quotient ring. For any ideal $B$ of $R$, let

$$O_l(B) = \{q \in Q\colon\ qB \subseteq B\}, \quad O_r(B) = \{q \in Q\colon\ Bq \subseteq B\}.$$

A finite sequence $M_1, M_2, \ldots, M_n$ of distinct idempotent maximal ideals of $R$ is called a *cycle of length n*, if $n > 1$, $O_r(M_i) = O_l(M_{i+1})$ for $1 \leqslant i \leqslant n-1$, and $O_r(M_n) = O_l(M_1)$ ([3]). Intersection of a cycle in $R$ is a maximal invertible ideal, and any maximal invertible ideal, which is not a maximal ideal, is an intersection of a cycle of idempotent maximal ideals in $R$. To avoid a trivial situation, by a cycle (of maximal ideals) in $R$, we shall understand a cycle of idempotent maximal ideals, or a singleton set $\{M\}$, where $M$ is an invertible maximal ideal of $R$. For any module $M_R$, $d(M_R)$ will denote its composition length, and for any subset $X$ of $M$, $\mathrm{ann}_R(X)$ (or simply $\mathrm{ann}(X)$) will denote the annihilator of $X$ in $R$. For a maximal invertible ideal $A$ of $R$, a module $M_R$ is said to be $A$-primary, if for each $x \in M$, $xA^k = 0$, for some $k$. For any non-zero ideal $B$ of $R$, $R/B$ is a generalized uniserial ring ([1]). For any module $M$ over a ring $S$, $E_S(M)$ (or simply $E(M)$) will denote its injective hull.

**2. Idempotent ideals.** Throughout $R$ is an (hnp)-ring. A finite sequence $M_1, M_2, \dots, M_k$ of distinct idempotent maximal ideals of $R$ is called a *subcycle* if $O_r(M_i) = O_l(M_{i+1})$, for $1 \leqslant i \leqslant k-1$, but it is not a cycle.

LEMMA 1. *Let $M_1, M_2, \dots, M_k$ be a finite sequence of distinct idempotent maximal ideals of $R$, such that $O_r(M_i) = O_l(M_{i+1})$ for $1 \leqslant i \leqslant k-1$. Then $R/M_k M_{k-1} \dots M_1$ has homogeneous right socle, and for some uniserial right $R$-module $uR$ of length $k$, $\operatorname{ann}(uR) = M_k M_{k-1} \dots M_1$. Further for any minimal right ideal $I$ of $R/M_k M_{k-1} \dots M_1$, $IM_1 = 0$.*

Proof. By Goodearl and Warfield ([5, Lemma 18]) there exists a uniserial module $uR$ with composition series

$$0 = X_0 < X_1 < X_2 \dots < X_k = uR$$

such that $\operatorname{ann}(X_i/X_{i-1}) = M_i$, $1 \leqslant i \leqslant k$.

Let $B = \operatorname{ann}(uR)$. Since $R/B$ is a generalized uniserial ring and $uR$ is a faithful uniserial $R/B$-module, $R/B$ must have homogeneous right socle and any minimal right ideal $I$ of $R/B$ is isomorphic to $X_1$. So $IM_1 = 0$. Clearly $M_k M_{k-1} \dots M_1 \subseteq B$. We write

$$R/M_k M_{k-1} \dots M_1 = \oplus \sum_{i=1}^{t} e_i \bar{R},$$

where $\bar{R} = R/M_k M_{k-1} \dots M_1$, and $e_i$ are orthogonal, primitive idempotents in $\bar{R}$. If $f_i$ is the natural image of $e_i$ in $R/B$, then $f_i$ are all orthogonal primitive idempotents in $R/B$. Now $f_i(R/B)$ is embeddable in $uR$ and hence $f_i(R/B) \approx X_l$ for some $l \geqslant 1$. Then $f_i(R/B)M_j = f_i(R/B)$ for all $j > l$. So

$$f_i(R/B)\, M_k M_{k-1} \dots M_{l+1} = f_i(R/B).$$

Then also

$$e_i \bar{R}\, M_k M_{k-1} \dots M_{l+1} = e_i \bar{R}.$$

If $M_k M_{k+1} \dots M_1 \neq B$, then for some $i$,

$$d(e_i \bar{R}) > d\big(f_i(R/B)\big).$$

However $f_i(R/B) \approx X_l$ gives

$$d\big(f_i(R/B)\big) = l.$$

Notice that for any uniserial $R$-module $yR$ and any non-zero maximal ideal $M$ of $R$, either $yM = yR$ or $yM$ is the unique maximal submodule of $yR$. Thus

$$\begin{aligned}(e_i \bar{R})\, M_k M_{k-1} \dots M_1 &= (e_i \bar{R})(M_k M_{k-1} \dots M_{l+1})(M_l M_{l-1} \dots M_1) \\ &= (e_i \bar{R})\, M_l M_{l-1} \dots M_1\end{aligned}$$

gives

$$d[(e_i\bar{R})M_kM_{k-1}\ldots M_1] \geqslant d(e_i\bar{R})-l>0.$$

But $\bar{R}M_kM_{k-1}\ldots M_1=0$, gives

$$e_i\bar{R}M_kM_{k-1}\ldots M_1=0.$$

This is a contradiction. Hence the result follows. □

LEMMA 2. *For any subcycle $M_1, M_2,\ldots, M_k$ in $R$, $B=M_kM_{k-1}\ldots M_1$ is an idempotent ideal, and*

$$B=(M_1\cap M_2\cap\ldots\cap M_k)^k \neq (M_1\cap M_2\cap\ldots\cap M_k)^{k-1}.$$

Proof. Since $M_1, M_2,\ldots, M_k$ do not contain a cycle, by [3, Proposition (4.5)], $A=(M_1\cap M_2\cap\ldots\cap M_k)^k$ is an idempotent ideal. Clearly $A\subseteq B$. Consider any uniserial summand $e\bar{R}$ of $\bar{R}=R/A$. Let its composition series be

$$e\bar{R}=X_t>X_{t-1}>\ldots>X_0=0.$$

For some $j$, $\operatorname{ann}(X_1)=M_j$. By [5, Theorem 8],

$$O_r(\operatorname{ann}_R(X_1))=O_l(\operatorname{ann}_R(X_2/X_1)).$$

So that $\operatorname{ann}_R(X_2/X_1)=M_{j+1}$. Continuing we get $\operatorname{ann}_R(X_t/X_{t-1})=M_{j+t-1}$, and hence $e\bar{R}M_kM_{k-1}\ldots M_1=0$. So that

$$B\subseteq \operatorname{ann}_R(\bar{R})=A.$$

This proves that $A=B$. Further as in Lemma 1 there exists a uniserial $R$-module $uR$ with composition series

$$uR=Y_k>Y_{k-1}>\ldots>Y_0=0$$

such that $M_i=\operatorname{ann}_R(Y_i/Y_{i-1})$.

For $C=M_1\cap M_2\cap\ldots\cap M_k$, we have $Y_{k-i}=uC^i$, $1\leqslant i\leqslant k$. Thus $uRC^{k-1}\neq 0$. Consequently

$$B=(M_1\cap M_2\cap\ldots\cap M_k)^k\neq(M_1\cap M_2\cap\ldots\cap M_k)^{k-1}. \ □$$

Definition 3. An idempotent ideal of the form $M_kM_{k-1}\ldots M_1$, where $M_1, M_2,\ldots, M_k$ is a subcycle in $R$, is called an *sc-idempotent ideal in $R$ given by* subcycle $M_1, M_2,\ldots, M_k$.

LEMMA 4. *Let $M$ and $N$ be two distinct idempotent maximal ideals of $R$ such that $O_r(M)\neq O_l(N)$, $O_r(N)\neq O_l(M)$. Then*

$$MN=NM=M\cap N.$$

Proof. For any two simple $R$-modules $S$ and $T$, with $SM=0=TN$ the hypothesis give $\operatorname{Ext}^1(S,T)=0=\operatorname{Ext}^1(T,S)$ (See [5, Theorem 8]). Thus

$R/(M \quad N)^2$ which is a generalized uniserial ring, must be a direct sum of two primary rings. So its maximal ideals $\bar{M} = M/(M \cap N)^2$, $\bar{N} = N/(M \cap N)^2$ commute. This gives $MN = NM = N \cap M$. □

LEMMA 5. *Let $M_1, M_2, \ldots, M_k$ and $N_1, N_2, \ldots, N_s$ be two disjoint subcycles in $R$, such that $O_r(M_k) \neq O_l(N_1)$ and $O_r(N_s) \neq O_l(M_1)$. Then the sc-idempotent ideals $B = M_k M_{k-1} \ldots M_1$, $C = N_s N_{s-1} \ldots N_1$ commute and are comaximal.*

Proof. The hypothesis yields $O_r(M_i) \neq O_l(N_j)$ and $O_r(N_j) \neq O_l(M_i)$ for all $i$, $j$. So by Lemma 4, $M_i N_j = N_j M_i$. Consequently $BC = CB$. □

Let now $A$ be any non-zero idempotent ideal of $R$. Let $\mathscr{F}$ be the family of those maximal ideals that contain $A$. Then $\mathscr{F}$ is a finite family and its members are idempotents. If $\mathscr{F}$ has $k$ members, say $M_1, M_2, \ldots, M_k$, by [3, Proposition (4.5)], $A = (\bigcap_{i=1}^{k} M_i)^k$. For any $M, N \in \mathscr{F}$, put $M \sim N$ if and only if there exist members $M = N_1, N_2, \ldots, N_u = N$ of $\mathscr{F}$ such that either $N_1, N_2, \ldots, N_u$ is a subcycle or $N_u, N_{u-1}, \ldots, N_1$ is a subcycle. This is an equivalence relation on $\mathscr{F}$. The members of an equivalence class can be arranged into a subcycle, so they determine an sc-idempotent ideal. By Lemma 5, such sc-idempotent ideals commute and are pairwise comaximal. Then appealing to Lemma 2 and [3, Proposition (4.5)] we get $A$ is the product of the sc-idempotent ideals determined by the equivalence classes in $\mathscr{F}$. This all gives the following.

THEOREM 6. *Let $A$ be a non-zero idempotent ideal of an (hnp)-ring $R$. Let $B_1, B_2, \ldots, B_s$ be the sc-idempotent ideals determined by maximal subcycle of idempotent maximals ideals containing $A$ and $t$ be the largest of lengths of such subcycles. Further let $M_1, M_2, \ldots, M_l$ be the totality of maximal ideals containing $A$. Then*

$$A = B_1 B_2 \ldots B_s = (M_1 \cap M_2 \cap \ldots \cap M_l)^t \neq (M_1 \cap M_2 \cap \ldots \cap M_l)^{t-1}.$$

**3. Over-rings.** Let $R$ be an (hnp)-ring and $Q$ be its classical quotient ring. Any subring $S$ of $Q$ containing $R$ is called an *over-ring of* $R$. Let $X$ be a collection of maximal right ideals of $R$. Let $\mathscr{S}_X$ be the set of all those essential right ideals $I$ of $R$, for which $R/I$ has no composition factor isomorphic to any of the modules $R/M$, where $M \in X$. This $\mathscr{S}_X$ is an additive topology on $R$, which determines a hereditary torsion theory on $R$ with corresponding quotient ring

$$R_X = \{x \in Q\colon\ xI \subseteq R \text{ for some } I \in \mathscr{S}_X\}$$

$R_X$ is called a *localization* of $R$ at $X$ ([4, p. 138]). Goodearl ([4, Theorem 5]) showed that every over-ring of $R$ is of the type $R_X$. Any simple $R_X$-module is of the type $R_X/KR_X$, $K \in X$ ([4, Theorem 5]). We shall use the above notations in succeeding results without comments. To avoid the trivial case, we assume that an over-ring $S$ does not equal $Q$.

LEMMA 7. *For any right ideal $A$ of an over-ring $S$ of an (hnp)-ring $R$, $A = (A \cap R)S$.*

Proof. Now $S = R_X$. Consider $x \in A$. For some $I \in \mathscr{S}_X$, $xI \subseteq R$. Then

$$I^* = \{ y \in Q \colon yI \subseteq R \} \subseteq R_X.$$

As $I_R$ is projective, $1 \in II^*$. Consequently $x \in xII^* \subseteq (A \cap R)S$. This proves that $A = (A \cap R)S$. □

LEMMA 8. *Consider any non-faithful simple $R_X$-module $R_X/KR_X$ with $M = \operatorname{ann}_{R_X}(R_X/KR_X)$. Then there exists a subcycle (or cycle) of maximal ideals $P_1, P_2, \ldots, P_t$ in $R$ such that $P_1 = \operatorname{ann}_R(R/K)$, $P_t P_{t-1} \ldots P_1 = \operatorname{ann}_R(R_X/KR_X)$, $R_X/KR_X$ is a uniserial $R$-module and*

$$M = (P_t P_{t-1} \ldots P_1) R_X.$$

Proof. Since $R/K$ is embeddable in $R_X/KR_X$ as an essential $R$-submodule ([4, Theorem 2]), $R_X/KR_X$ is a uniform $R$-module. Further $M \neq 0$ gives $M \cap R \neq 0$. Consequently $R_X/KR_X$ is a uniform non-faithful $R$-module, and it is uniserial as $R$-module. Let its composition series as $R$-module be

$$R_X/KR_X = X_t > X_{t-1} > \ldots > X_1 > X_0 = 0.$$

Let $P_i = \operatorname{ann}_R(X_i/X_{i-1})$, $1 \leqslant i \leqslant t$. Then $P_1 = \operatorname{ann}_R(R/K)$. As seen in the proof of [4, Theorem 3] $R/K$ is $\mathscr{S}_X$-torsionfree, but $X_t/X_1$ is $\mathscr{S}_X$-torsion. Consequently because of periodicity theorem, no two $P_i$'s can be equal. By Lemma 1,

$$\operatorname{ann}_R(R_X/KR_X) = P_t P_{t-1} \ldots P_1.$$

Then Lemma 7 yields,

$$M = (P_t P_{t-1} \ldots P_1) R_X. \ \square$$

THEOREM 9. *Let $M$ be a non-zero maximal ideal of an over-ring $R_X$ of an (hnp)-ring $R$, and $R_X/KR_X$ be a simple $R_X$-module with $M = \operatorname{ann}_{R_X}(R_X/KR_X)$ and $P = \operatorname{ann}_R(R/K)$. Then $M$ belongs to a cycle of maximal ideals in $R_X$ if and only if $P$ belongs to a cycle of maximal ideals in $R$.*

Proof. Consider $E = E_R(R/K)$. As $R/K$ is $\mathscr{S}_X$-torsionfree, $E$ is also injective as $R_X$-module, by [7, Lemma (2.6), p. 202] and $R_X/KR_X$ is embeddable in $E$ as $R_X$-module. Let $M = M_1, M_2, \ldots, M_t$ be a cycle in $R_X$. By [5, Lemma 18], $E$ has a uniserial $R_X$-submodule $uR_X$ of length $t+1$, such that for its composition series

$$uR_X = Y_{t+1} > Y_t > \ldots > Y_1 > Y_0 = 0,$$

$$M_i = \operatorname{ann}_{R_X}(Y_i/Y_{i-1}), \qquad 1 \leqslant i \leqslant t,$$

$$M_1 = \operatorname{ann}_{R_X}(Y_{t+1}/Y_t).$$

Then $uR_X$ being also non-faithful, uniform $R$-module, is uniserial as $R$-module. Further $Y_{t+1}/Y_t \approx Y_1$ as $R_X$-modules, give $Y_{t+1}/Y_t$ and $Y_1$ have isomorphic $R$-socles. This gives $uR_X$ have some isomorphic $R$-composition factors. Consequently $P = \operatorname{ann}_R(R/K)$ belongs to a cycle in $R$. Conversely let $P$ belong to a cycle in $R$. Then any proper $R$-submodule of $E$ is uniserial. Consequently every proper $R_X$-submodule of $E$ is also uniserial and non-faithful. Since $(R_X/KR_X)/(R/K)$ is $\mathscr{S}_X$-torsion, using the periodicity theorem, we get $E$ is of infinite length as $R_X$-module, and hence by [5, Theorem 20], $M$ belongs to a cycle in $R_X$. □

Similar arguments also give the following.

THEOREM 10. *Let a prime ideal $P$ of an (hnp)-ring $R$ belong to a cycle in $R$. For any over-ring $R_X$ of $R$, either $PR_X = S$ or for some $K \in X$, $P = \operatorname{ann}(R/K)$, with $M = \operatorname{ann}_{R_X}(R_X/KR_X)$ a maximal ideal belonging a cycle in $S$.*

THEOREM 11. *Any over-ring of an (hnp)-ring $R$ with enough invertible ideals, has enough invertible ideals.*

Proof. Let $S = R_X$ be any over-ring of $R$. Let $M$ be any non-zero idempotent, maximal ideal of $S$. Let $R_X/KR_X$ be a simple $R_X$-module with $M = \operatorname{ann}_{R_X}(R_X/KR_X)$. Let $P = \operatorname{ann}_R(R/K)$. Since $R$ has enough invertible ideals, $P$ belongs to a cycle in $R$ ([3, Corollary (4.7)]). Consequently by Theorem 8, $M$ belongs to a cycle in $S$, and hence by [3, Corollary (4.7)], $S$ has enough invertible ideals. □

LEMMA 12. *Let $M_1, M_2, \ldots, M_t$ be a subcycle (cycle) of maximal ideals in an over-ring $S$ of an (hnp)-ring $R$. There exists a subcycle (cycle) $N_1, N_2, \ldots, N_u$ of maximal ideals in $R$ and positive integers $1 = k_1 < k_2 < \ldots \ldots < k_s \leqslant u$ with*

$$M_1 = (N_{k_2-1} \ldots N_1)S, \quad M_2 = (N_{k_3-1} \ldots N_{k_2})S, \ldots,$$
$$M_t = (N_u N_{u-1} \ldots N_{k_t})S.$$

*Further $M_t M_{t-1} \ldots M_1 \cap R = N_u N_{u-1} \ldots N_1$.*

Proof. There exists a uniserial $R_X$-module $Y$ with composition series

$$Y = Y_t > Y_{t-1} > \ldots > Y_0 = 0$$

with $M_i = \operatorname{ann}(Y_i/Y_{i-1})$ ([5, Lemma 18]). By Lemma 8, $\operatorname{ann}_R(Y_i/Y_{i-1}) \neq (0)$. So that $Y$ is a non-faithful $R$-module and hence $Y_R$ being uniform yields $Y$ is a uniserial $R$-module of length say $u$. Let

$$Y = X_u > X_{u-1} > \ldots > X_0 = 0$$

be the $R$-composition series of $Y$ and $N_i = \operatorname{ann}(X_i/X_{i-1})$, $1 \leqslant i \leqslant u$. Let for some $v$, $w$, $1 \leqslant v < w \leqslant u$, $N_v = N_w$.

Now, $S = R_X$, and $\operatorname{socle}_R(Y_i/Y_{i-1})$ are $\mathcal{S}_X$-torsionfree and any proper $R$-homomorphic image of $Y_i/Y_{i-1}$ is $\mathcal{S}_X$-torsion. So there exists $j > k$, such that

$$\operatorname{socle}_R(Y_j/Y_{j-1}) \approx \operatorname{socle}(Y_k/Y_{k-1}).$$

This yields $M_j = M_k$. This is a contradiction. Hence $N_1, N_2, \ldots, N_u$ are all distinct. Consequently either $u = 1$ or $N_i$ are idempotent ideals, and $O_r(N_i) = O_l(N_{i+1})$. If $u = 1$, then $t = 1$. It can be easily seen, by following the arguments in Theorem 9 that $M_1, M_2, \ldots, M_t$ is a cycle if and only if $N_1, N_2, \ldots, N_u$ is a cycle. This proves the lemma. □

The above lemma gives the following.

COROLLARY 13. *Let $M$ be a simple module over an (hnp)-ring $R$, such that $P = \operatorname{ann}_R(M)$ is an invertible maximal ideal in $R$. Then for any over-ring $S$ of $R$, either $M \otimes_R S = 0$ or $(M \otimes S)_R \approx M_R$.*

Let $A$ be an idempotent non-zero ideal in an (hnp)-ring $R$. Eisenbud and Robson ([3, Proposition (1.8)]) showed that $B \leftrightarrow BA$, $B$ being an idempotent ideal of $O_l(A)$ is a one-to-one correspondence between idempotent ideals of $O_l(A)$ and those idempotent ideals of $R$ that are contained in $A$. The next theorem shows that for any overring $S$ of $R$, there is a one-to-one correspondence between idempotent ideals of $S$, and certain family of idempotent ideals of $R$.

THEOREM 14. *For any idempotent ideal $A$ of an over-ring $S$ of an (hnp)-ring $R$, $A \cap R$ is an idempotent ideal of $R$.*

Proof. It follows from Lemma 12 that if $A$ is an sc-idempotent ideal of $S$, then $A \cap R$ is an sc-idempotent ideal of $R$. In general let $A$ be a non-zero idempotent ideal of $S$. By Theorem 6, $A = B_1 B_2 \ldots B_s$ for some mutually commuting pairwise co-maximal sc-idempotent ideals $B_i$ of $S$, such that the subcycles determining these $B_i$'s are pairwise disjoint, and union of no two of these subcycles is a subcycle. By Lemma 12, $B_i \cap R$ is an sc-idempotent ideal of $R$. It can be easily seen that the subcycles determining these idempotent ideals $B_i \cap R$ are also pairwise disjoint, and union of no two such subcycles is a subcycle. Consequently $B_i \cap R$ are pairwise comaximal and they mutually commute. Hence $\prod_i (B_i \cap R)$ is an idempotent ideal of $R$. Clearly

$$A \cap R = \Big(\prod_i B_i\Big) \cap R = \Big(\bigcap_i B_i\Big) \cap R = \bigcap_i (B_i \cap R) = \prod_i (B_i \cap R).$$

This proves the theorem. □

Since an (hnp)-ring $R$ is a Dedekind prime ring if and only if $R$ has no idempotent ideal ([3, Theorem (1.2)]) the above theorem immediately gives the following well-known result.

COROLLARY 15. *Any over-ring of a Dedekind prime ring is a Dedekind prime ring.*

Remark. The correspondence given by Theorem 14, does not extend that given by Eisenbud and Robson ([3, Proposition (1.8)]).

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