COLLOQUIUM MATHEMATICUM

VOL. XLIII 1980 FASC. 1

HOMOGENEOUS SUPERASSOCIATIVE SYSTEMS

BY

H. LÄNGER (VIENNA)

The notion of homogeneity was introduced by Marczewski in [4]. *Homogeneous algebras* are exactly those algebras whose automorphism group is the whole symmetric group. Ganter et al. have proved that all the non-trivial homogeneous algebras are simple with one single exception (cf. [2]). In the following we shall investigate homogeneity within the variety of superassociative systems. These systems are quite natural generalizations of semigroups. A *superassociative system* is an algebra with one $(n+1)$-ary operation $(n \geqslant 1)$ satisfying some law which in case $n = 1$ reduces to the well-known associative law. Superassociativity turns out to be the essential property of composition of functions since for superassociative systems there holds some sort of Cayley-representation theorem generalizing that one valid for semigroups. The concept of superassociativity has its origin in works of Menger (cf. [5] and [6]). In [1] Dicker proved that any superassociative system can be embedded into the full function algebra over some suitable set, a result which was proved in a more general version by Nöbauer in [7]. Some material concerning superassociative systems can also be found in a book by Lausch and Nöbauer (cf. [3], Chapter 3). Our aim in this note is to give a classification of all homogeneous superassociative operations on some fixed set.

In the sequel let A be some fixed set.

Definition 1. Let n be some positive integer and let $f \in A^{A^{n+1}}$. The operation f is called *homogeneous* if

$$gf(x_0, \ldots, x_n) = f(gx_0, \ldots, gx_n)$$

for any $x_0, \ldots, x_n \in A$ and for any $g \in \operatorname{Sym} A$ (i.e. symmetric group over A).

The operation f is called *superassociative* if

$$f\big(f(x_0, \ldots, x_n), x_{n+1}, \ldots, x_{2n}\big) = f\big(x_0, f(x_1, x_{n+1}, \ldots, x_{2n}), \ldots, f(x_n, x_{n+1}, \ldots, x_{2n})\big)$$

for any $x_0, \ldots, x_{2n} \in A$.

In the following let n be some fixed positive integer and let f be some fixed $(n+1)$-ary homogeneous superassociative operation on A.

LEMMA 1. *Assume that* $a_0, \ldots, a_n \in A$ *and* $f(a_0, \ldots, a_n) \neq a_0, \ldots, a_n$. *Then*

$$A = \{a_0, \ldots, a_n, f(a_0, \ldots, a_n)\}.$$

Proof. Suppose that $A \neq \{a_0, \ldots, a_n, f(a_0, \ldots, a_n)\}$. Let

$$a \in A \setminus \{a_0, \ldots, a_n, f(a_0, \ldots, a_n)\}$$

and put

$$g := \big(a\ f(a_0, \ldots, a_n)\big).$$

Then

$$a = gf(a_0, \ldots, a_n) = f(ga_0, \ldots, ga_n) = f(a_0, \ldots, a_n) \neq a,$$

a contradiction.

LEMMA 2. $f(x, \ldots, x) = x$ *for any* $x \in A$.

Proof. Suppose there exist some $a \in A$ such that $f(a, \ldots, a) \neq a$. Then $A = \{a, f(a, \ldots, a)\}$ by Lemma 1. Put $g := \big(a\ f(a, \ldots, a)\big)$. Then

$$\begin{aligned} f\big(f(a, \ldots, a), a, \ldots, a\big) &\neq gf\big(f(a, \ldots, a), a, \ldots, a\big) \\ &= f\big(gf(a, \ldots, a), ga, \ldots, ga\big) = f\big(a, f(a, \ldots, a), \ldots, f(a, \ldots, a)\big) \\ &= f\big(f(a, \ldots, a), a, \ldots, a\big), \end{aligned}$$

a contradiction.

LEMMA 3. *Assume that* $a_0, \ldots, a_n \in A$ *and* $f(a_0, \ldots, a_n) \neq a_0$. *Then* $f(x, y, \ldots, y) = y$ *for any* $x, y \in A$.

Proof. Let $a, b \in A$. If $a = b$, then $f(a, b, \ldots, b) = f(b, \ldots, b) = b$ by Lemma 2. Now assume that $a \neq b$. Let $g \in \operatorname{Sym} A$ be such that $ga_0 = a$ and $gf(a_0, \ldots, a_n) = b$. Then, using Lemma 2, one obtains

$$\begin{aligned} f(a, b, \ldots, b) &= f\big(ga_0, gf(a_0, \ldots, a_n), \ldots, gf(a_0, \ldots, a_n)\big) \\ &= gf\big(a_0, f(a_0, \ldots, a_n), \ldots, f(a_0, \ldots, a_n)\big) \\ &= gf\big(f(a_0, \ldots, a_0), a_1, \ldots, a_n\big) = gf(a_0, \ldots, a_n) = b. \end{aligned}$$

Hence, in any case, $f(a, b, \ldots, b) = b$. This completes the proof of the lemma.

Definition 2. For any $(a_1, \ldots, a_n) \in A^n$ define $f_{a_1, \ldots, a_n} \in A^A$ by

$$f_{a_1, \ldots, a_n} x := f(x, a_1, \ldots, a_n)$$

for any $x \in A$.

LEMMA 4. *Assume that* $a_1, \ldots, a_n \in A$, *let* $g \in \operatorname{Sym} A$ *and put* $h := f_{a_1,\ldots,a_n}$. *Then*

$$f_{ga_1,\ldots,ga_n} = ghg^{-1} \tag{*}$$

and

$$f_{hga_1,\ldots,hga_n} = hghg^{-1}. \tag{**}$$

The proof is trivial.

LEMMA 5. *Assume that* $a_1, \ldots, a_n \in A$. *Then*

$$f_{a_1,\ldots,a_n} = 1_A \quad \textit{or} \quad |f_{a_1,\ldots,a_n}A| = 1.$$

Proof. We use induction on n. From Lemma 3 it follows that Lemma 5 is true for $n = 1$. Now let $n > 1$ and assume that Lemma 5 has already been proved for $n-1$ instead of n. Put

$$B := \{a_1, \ldots, a_n\} \quad \text{and} \quad h := f_{a_1,\ldots,a_n}.$$

Suppose $h \neq 1_A$ and $|hA| \neq 1$. If $b_1, \ldots, b_n \in A$ and $|\{b_1, \ldots, b_n\}| < n$, say $b_1 = b_n$, and if one defines $f_1 \in A^{A^n}$ by

$$f_1(x_0, \ldots, x_{n-1}) := f(x_0, \ldots, x_{n-1}, x_1)$$

for any $x_0, \ldots, x_{n-1} \in A$, then f_1 is homogeneous and superassociative and

$$f_{b_1,\ldots,b_n}x = f_1(x, b_1, \ldots, b_{n-1})$$

for any $x \in A$, whence, by induction hypothesis, $f_{b_1,\ldots,b_n} = 1_A$ or $|f_{b_1,\ldots,b_n}A| = 1$. Hence, whenever $b_1, \ldots, b_n \in A$ are such that $|\{b_1, \ldots, b_n\}| < n$, then $f_{b_1,\ldots,b_n} = 1_A$ or $|f_{b_1,\ldots,b_n}A| = 1$. Therefore, $|B| < n$ would imply $h = 1_A$ or $|hA| = 1$ both contradicting our assumption. Hence $|B| = n$. Now $h \in \operatorname{Sym} A$ would imply

$$h = hh^{-1}hhh^{-1} = f_{hh^{-1}a_1,\ldots,hh^{-1}a_n}h^{-1} = hh^{-1} = 1_A$$

(by (**)) contradicting our assumption. Hence $h \notin \operatorname{Sym} A$. If $|hB| < n$, then, because of $h^2 = f_{ha_1,\ldots,ha_n}$ (by (**)), we have $h^2 = 1_A$ or $|h^2A| = 1$, the first being impossible since $h \notin \operatorname{Sym} A$. Hence, if $|hB| < n$, then $|h^2A| = 1$. Now let $g \in \operatorname{Sym} A$ be such that

(i) $gx \in h^{-1}\{x\} \cap B$ if both $x \in hA$ and $h^{-1}\{x\} \cap B \neq \emptyset$;

(ii) $gx \in h^{-1}\{x\}$ if both $x \in hA$ and $h^{-1}\{x\} \cap B = \emptyset$;

(iii) $gB = B$ if both $h^{-1}\{x\} \cap B \neq \emptyset$ for any $x \in hA \cap B$ and $h^{-1}\{x\} \cap B = \emptyset$ for any $x \in hA \setminus B$.

Then $hgh = h$. If $|hgB| < n$, then, because of

$$hg^{-1} = hghg^{-1} = f_{hga_1,\ldots,hga_n}$$

(by (**)), we would have $hg^{-1} = 1_A$ or $|hg^{-1}A| = 1$, the first implying

$h = g \in \operatorname{Sym} A$ and the second implying $|hA| = 1$, both being a contradiction. Hence $|hgB| = n$. Now we consider the following cases:

Case 1. $hB \subseteq B$ and $h^{-1}\{x\} \cap B \neq \emptyset$ for any $x \in hA \cap B$.

Then $h^{-1}\{x\} \cap B = \emptyset$ for any $x \in hA \setminus B$ and, therefore, $gB = B$, whence $g|B \in \operatorname{Sym} B$. Since $hB \subseteq B$ and $|hB| = |hgB| = n = |B|$, we have $hB = B$, whence $h|B \in \operatorname{Sym} B$. Now

$$h|B = hghg^{-1}g|B = f_{hga_1,\ldots,hga_n}g|B = hg|B = (hgh|B)(h|B)^{-1}$$
$$= (h|B)(h|B)^{-1} = 1_B$$

(by (**)). Since $h \neq 1_A$, there exists some $a \in A$ such that $ha \neq a$. Because of $h|B = 1_B$ we have $a \in A \setminus B$. Let $b \in B \setminus \{ha\}$ and put

$$g_1 := (a\ ha\ b) \quad \text{and} \quad g_2 := (ha\ b).$$

Then

$$b = hg_1hg_1^{-1}ha = f_{hg_1a_1,\ldots,hg_1a_n}ha = f_{g_2a_1,\ldots,g_2a_n}ha = g_2hg_2ha = ha \neq b$$

(by (**) and (*)), a contradiction.

Case 2. $hB \subseteq B$ and there exists some $x \in hA \cap B$ such that $h^{-1}\{x\} \cap B = \emptyset$.

Let $c \in hA \cap B$ be such that $h^{-1}\{c\} \cap B = \emptyset$. Then $hB \subseteq B \setminus \{c\}$ and, therefore, $|hB| < n$, whence $|h^2A| = 1$. Let $d \in h^{-1}\{c\}$. Then $d \in A \setminus B$. If $e \in A \setminus B$ and $g_3 \in \operatorname{Sym} A$ are such that $g_3|B = 1_B$ and $g_3d = e$, then

$$he = f(e, a_1, \ldots, a_n) = f(g_3d, g_3a_1, \ldots, g_3a_n) = g_3f(d, a_1, \ldots, a_n)$$
$$= g_3hd = g_3c = c.$$

Hence $h(A \setminus B) = \{c\}$. Now

$$n-1 = |hgB|-1 \leqslant |hA|-1 = |hB \cup h(A \setminus B)|-1 \leqslant |hB| \leqslant |B \setminus \{c\}| = n-1,$$

whence $|hB| = |B \setminus \{c\}|$ which together with $hB \subseteq B \setminus \{c\}$ implies $hB = B \setminus \{c\}$. Now we conclude

$$n = |B \setminus \{c\}|+1 = |hB|+1 = \big|h(hB \cup h(A \setminus B))\big|+1$$
$$= \big|h^2(B \cup (A \setminus B))\big|+1 = |h^2A|+1 = 2.$$

Put $g_4 := (c\ d)$. Then

$$hc = f_{hg_4a_1,hg_4a_2}c = hg_4hg_4c = c$$

(by (**)), whence $c \in h^{-1}\{c\} \cap B = \emptyset$, a contradiction.

Case 3. $hB \nsubseteq B$.

Then there exists some $e_1 \in B$ such that $he_1 \in A \setminus B$. We have

$$f(e_1, a_1, \ldots, a_n) = he_1 \neq e_1, a_1, \ldots, a_n,$$

whence $A = B\cup\{he_1\}$ by Lemma 1. Therefore, $|A| = n+1$. Since $h \notin \mathrm{Sym}\,A$, we have $|hA| \leqslant n = |hgB| \leqslant |hA|$, whence $|hA| = n$. Thus $|A| = n+1$ and $|hA| = n$ together imply $|hA|-1 \leqslant |h(hA)|$. Now suppose that $|hB| < n$. Then $|h^2A| = 1$ and, therefore,

$$2 \leqslant n = |hA| \leqslant |h^2A|+1 = 2,$$

whence $n = 2$. Hence $|hB| = 1$ and, therefore, $hB = \{he_1\}$. Now we conclude

$$2 = n = |hA| = |hB\cup h\{he_1\}| = |\{he_1, h^2e_1\}| \leqslant 2,$$

whence $|\{he_1, h^2e_1\}| = 2$. Hence $h^2e_1 \in A\setminus\{he_1\} = B$. But now

$$\begin{aligned}1 &= |h^2A| = |h^2B\cup h^2\{he_1\}| = |h(hB)\cup h\{h^2e_1\}| = |h\{he_1\}\cup\{he_1\}|\\ &= |h\{he_1\}\cup hB| = |hA| = 2 \neq 1,\end{aligned}$$

a contradiction. Therefore, $|hB| = n$, whence $h|B = g^{-1}|B$. Now we conclude

$$\begin{aligned}A &= g^{-1}A = g^{-1}B\cup g^{-1}\{he_1\} = hB\cup\{g^{-1}hgg^{-1}e_1\}\\ &\subseteq hA\cup\{f_{g^{-1}a_1,\ldots,g^{-1}a_n}g^{-1}e_1\} = hA\cup\{f_{ha_1,\ldots,ha_n}g^{-1}e_1\}\\ &= hA\cup\{h^2g^{-1}e_1\} \subseteq hA \subseteq A\end{aligned}$$

(by (*) and (**)), whence $hA = A$ which together with $|A| = n+1$ implies $h \in \mathrm{Sym}\,A$, a contradiction.

Hence, in any of the three cases we obtain a contradiction. Therefore, $h = 1_A$ or $|hA| = 1$. Applying induction argument completes the proof of the lemma.

THEOREM. *Let A be some set, let n be some positive integer, and let f be some $(n+1)$-ary homogeneous (not necessarily superassociative) operation on A. Then the following conditions are equivalent:*

(i) *f is superassociative.*

(ii) (A) *or* (B):

(A) *$f(x_0, \ldots, x_n) = x_0$ for any $x_0, \ldots, x_n \in A$.*

(B) (a)-(c):

(a) *If $x_1, \ldots, x_n \in A$ and $|\{x_1, \ldots, x_n\}| = 1$, then $f(x, x_1, \ldots, x_n) = x_1$ for any $x \in A$.*

(b) *If $x_1, \ldots, x_n \in A$, $|\{x_1, \ldots, x_n\}| > 1$ and $|\{x_1, \ldots, x_n\}|+1 \neq |A|$, then* ($\alpha$) *or* ($\beta$):

(α) *$f(x, x_1, \ldots, x_n) = x$ for any $x \in A$.*

(β) *There exists some integer i, $1 \leqslant i \leqslant n$, such that $f(x, x_1, \ldots, x_n) = x_i$ for any $x \in A$.*

(c) *If $x_1, \ldots, x_n \in A$, $|\{x_1, \ldots, x_n\}| > 1$ and $|\{x_1, \ldots, x_n\}|+1 = |A|$, then* ($\alpha$) *or* ($\beta$) *or* ($\gamma$):

(γ) *For any* $x \in A$ *the element* $f(x, x_1, \ldots, x_n)$ *is the unique element of* $A \setminus \{x_1, \ldots, x_n\}$.

Proof. Without loss of generality, $|A| > 1$. First assume (i). Suppose that (A) does not hold. Then (a) holds because of Lemma 3. Now let $x_1, \ldots, x_n \in A$ and suppose that neither (α) nor (β) holds. Then $|\{x_1, \ldots, x_n\}| > 1$ by Lemma 3. Because of Lemma 5 there exists some $a \in A \setminus \{x_1, \ldots, x_n\}$ such that $f(x, x_1, \ldots, x_n) = a$ for any $x \in A$. Hence $f(x_1, x_1, \ldots, x_n) = a$, and thus $A = \{a, x_1, \ldots, x_n\}$ by Lemma 1. Therefore, $|\{x_1, \ldots, x_n\}| + 1 = |A|$, whence ($\gamma$). Hence (B) holds, and thus (ii) is proved. Since it is easy to see that (ii) implies (i), the proof of the theorem is completed.

REFERENCES

[1] R. M. Dicker, *The substitutive law*, Proceedings of the London Mathematical Society 13, Series 3 (1963), p. 493-510.

[2] B. Ganter, J. Płonka and H. Werner, *Homogeneous algebras are simple*, Fundamenta Mathematicae 79 (1973), p. 217-220.

[3] H. Lausch and W. Nöbauer, *Algebra of polynomials*, Amsterdam 1973.

[4] E. Marczewski, *Homogeneous operations and homogeneous algebras*, Fundamenta Mathematicae 56 (1964), p. 81-103.

[5] K. Menger, *Algebra of analysis*, Notre Dame Mathematical Lectures 3 (1944).

[6] — *Superassociative systems and logical functors*, Mathematische Annalen 157 (1964), p. 278-295.

[7] W. Nöbauer, *Über die Darstellung von universellen Algebren durch Funktionenalgebren*, Publicationes Mathematicae (Debrecen) 10 (1963), p. 151-154.

TECHNISCHE UNIVERSITÄT WIEN
INSTITUT FÜR ALGEBRA UND MATHEMATISCHE STRUKTURTHEORIE
VIENNA

Reçu par la Rédaction le 4. 10. 1977;
en version modifiée le 30. 5. 1978