COLLOQUIUM MATHEMATICUM

VOL. XXVIII 1973 FASC. 1

VARIETIES OF TOPOLOGICAL GROUPS GENERATED BY MAXIMALLY ALMOST PERIODIC GROUPS

BY

SIDNEY A. MORRIS (SYDNEY)

It was shown by Brooks, Saxon and the author [1] that if Ω is a class of locally compact abelian groups or a class of compact (indeed, pseudocompact) groups, then every Hausdorff group in the variety of topological groups $\mathbf{V}(\Omega)$ generated by Ω is *maximally almost periodic* (MAP). It is natural to ask:

If Ω is a class of locally compact MAP groups, is every Hausdorff group in $\mathbf{V}(\Omega)$ (necessarily) MAP?

The question is answered in the negative. However, an affirmative answer is obtained if G is also assumed to be connected.

Preliminaries. A non-empty class $\mathbf{V}$ of (not necessarily Hausdorff) topological groups is said to be a *variety* if it is closed under the operations of taking subgroups, quotient groups, arbitrary cartesian products and isomorphic images (see [1], [2] and [9]-[14]). The smallest variety containing a class Ω of topological groups is said to be the *variety generated by* Ω and it is denoted by $\mathbf{V}(\Omega)$.

It Ω is a class of topological groups, then $\mathbf{S}(\Omega)$ denotes the class of all topological groups isomorphic to subgroups of members of Ω. Similarly, we define the operators $\bar{\mathbf{S}}, \bar{\mathbf{Q}}$, $\mathbf{C}$ and $\mathbf{D}$, where they denote closed subgroup, separated quotient, arbitrary cartesian product and finite product, respectively.

THEOREM 0.1 [1]. *If Ω is a class of topological groups and G is a Hausdorff group in* $\mathbf{V}(\Omega)$, *then* $G \in \mathbf{S}\mathbf{C}\bar{\mathbf{Q}}\bar{\mathbf{S}}\mathbf{D}(\Omega)$.

A topological group G is said to be *maximally almost periodic* (MAP) [15] if there exists a continuous monomorphism of G into a compact group. Locally compact abelian groups and compact (even pseudocompact [3]) groups are examples of MAP groups.

A topological group G is said to be an SIN *group* [6] if each neighbourhood of the identity contains a neighbourhood of the identity invariant under the inner automorphisms of G. Abelian topological groups, discrete groups and compact groups provide examples of SIN groups.

A topological group G is said to be a Z *group* [6] if the quotient group $G/Z(G)$ is compact, where $Z(G)$ is the centre of G. As examples of Z groups we have abelian topological groups and compact groups.

Whenever the assumption of local compactness is added we speak of [MAP], [SIN] and [Z] groups. It is noted in [6] that [Z] implies both [MAP] and [SIN] and that no other implication between any pair of these is valid.

THEOREM 0.2 [6]. *If G is a connected topological group, then the following are equivalent*:

(i) *G is* [MAP];

(ii) *G is* [SIN];

(iii) *G is* [Z];

(iv) *G is isomorphic to $V \times L$, where V is a vector group and L is a compact group.*

Finally, we introduce Graev's concept [5] of a free topological group. Let X be a topological space with distinguished point e. Then the group $F(X)$ is said to be a *free topological group on X* if it has the following properties:

(a) X is a subspace of $F(X)$,

(b) X generates $F(X)$ algebraically, and e is the identity element of $F(X)$,

(c) for any continuous mapping φ of X into any topological group H such that $\varphi(e)$ is the identity element of H, there exists a continuous homomorphism Φ of $F(X)$ into H such that $\Phi | X = \varphi$.

Using arguments similar to those in [4], [5], [8] and [14], we obtain

THEOREM 0.3. *If X is any Tychonoff space, then*

(a) *$F(X)$ exists*,

(b) *$F(X)$ is unique (up to isomorphism)*,

(c) *$F(X)$ is* MAP,

(d) *if X is a topological group, then it is isomorphic to a quotient group of $F(X)$.*

Results. Our first example shows that a variety generated by [MAP] groups can contain Hausdorff non-MAP groups.

Example 1.1. Let Ω be the class of all discrete free topological groups. From Theorem 0.3, each member of Ω is MAP. To see that $\mathbf{V}(\Omega)$ contains Hausdorff non-MAP groups, we only have to note that (by Theorem 0.3) every discrete group is in $\mathbf{V}(\Omega)$ and that there exist [6] discrete non-MAP groups.

THEOREM 1.2. *If Ω is a class of* [MAP] *groups, then every connected Hausdorff group G in* $\mathbf{V}(\Omega)$ *is* MAP.

Proof. By Theorem 0.1, $G \in \mathbf{SC\overline{Q}\overline{S}D}(\Omega)$, that is, G is a subgroup of a product $\prod_{i \in I} G_i$, where each $G_i \in \mathbf{\overline{Q}\overline{S}D}(\Omega)$. Let H_i be the closure in G_i of the projection of G into G_i. Then $G \in \mathbf{SC}\{H_i : i \in I\}$ and each H_i is a connected locally compact group in $\mathbf{\overline{S}\overline{Q}\overline{S}D}(\Omega) \subset \mathbf{\overline{Q}\overline{S}D}(\Omega)$. In view of Theorem 0.2, to show that G is MAP it is enough to prove that every connected locally compact group H in $\mathbf{\overline{Q}\overline{S}D}(\Omega)$ is a [Z] group.

We note that each member of $\mathbf{\overline{S}D}(\Omega)$ is a [MAP] group. Thus, we only have to show that if A is a [MAP] group and H is a connected locally compact group in $\mathbf{\overline{Q}}\{A\}$, then H is a [Z] group. (Without the connectedness restriction on H, Example 1.1 shows that this is not necessarily true.)

Let K be the component of the identity in A. Then, by Theorem 0.2, K is a [Z] group. If f is the quotient mapping of A onto H, then, by Theorem 7.12 of [7], $f(K)$ is dense in H.

To complete the proof, we observe the following readily verified facts:

Let P and Q be Hausdorff groups with P a Z group. If (i) P is a dense subgroup of Q or (ii) Q is a continuous homomorphic image of P, then Q is a Z group.

Thus, in the present case, K a [Z] group implies $f(K)$ is a Z group which shows that H is a [Z] group, as required.

Our next example shows that without the assumption of local compactness on Ω, Theorem 1.2 would be false.

Example 1.3. Let G be any connected solvable non-abelian Lie group. Then, by Theorem 0.2, G is an arcwise connected locally compact non-MAP group. If $F(G)$ is the free topological group on G, then it is an arcwise connected MAP group which has G as a quotient group. Thus $\mathbf{V}(F(G))$ is a variety generated by an arcwise connected MAP group which contains a locally compact connected non-MAP group. (In light of this, it is interesting to note [14] that every free topological group of the variety $\mathbf{V}(F(G))$ is MAP.)

THEOREM 1.4. *If Ω is a class of* SIN *groups, then every member of* $\mathbf{V}(\Omega)$ *is an* SIN *group.*

Proof. This result follows immediately from the fact that subgroups, quotient groups and cartesian products of SIN groups are SIN groups.

Our final result should be contrasted with Example 1.3.

COROLLARY 1.5. *If Ω is a class of* SIN *groups, then every connected locally compact group in* $\mathbf{V}(\Omega)$ *is* MAP.

Proof. This is a consequence of Theorems 0.2 and 1.4.

REFERENCES

[1] M. S. Brooks, S. A. Morris and S. A. Saxon, *Generating varieties of topological groups*, Proceedings of the Edinburgh Mathematical Society 18 (1973), p. 191 - 197.

[2] Su-shing Chen and S. A. Morris, *Varieties of topological groups generated by Lie groups*, ibidem 18 (1972), p. 49 - 53.

[3] W. W. Comfort and K. A. Ross, *Pseudocompactness and uniform continuity in topological groups*, Pacific Journal of Mathematics 16 (1966), p. 483 - 496.

[4] B. R. Gelbaum, *Free topological groups*, Proceedings of the American Mathematical Society 12 (1961, p. 737 - 743.

[5] М. И. Граев, *Свободные топологические группы*, Известия Академии Наук, серия математическая, 12 (1948), p. 279 - 324. English translation in American Mathematical Society Translation 35 (1951), 61 pp. Reprint, ibidem (1) 8 (1962), p. 305 - 364.

[6] S. Grosser and M. Moskowitz, *Compactness conditions in topological groups*, Journal für die Reine und Angewandte Mathematik 246 (1971), p. 1 - 40.

[7] E. Hewitt and K. A. Ross, *Abstract harmonic analysis I*, Berlin 1963.

[8] S. Kakutani, *Free topological groups and infinite direct product topological groups*, Proceedings of the Imperial Academy of Tokyo 20 (1944), p. 595 - 598.

[9] S. A. Morris, *Varieties of topological groups*, Bulletin of the Australian Mathematical Society 1 (1969), p. 145 - 160.

[10] — *Varieties of topological groups II*, ibidem 2 (1970), p. 1 - 13.

[11] — *Varieties of topological groups III*, ibidem 2 (1970), p. 165 - 178.

[12] — *On varieties of topological groups generated by solvable groups*, Colloquium Mathematicum 25 (1972), p. 225 - 226.

[13] — *Locally compact abelian groups and the variety of topological groups generated by the reals*, Proceedings of the American Mathematical Society 34 (1972), p. 290 - 292.

[14] — *Maximally almost periodic groups and varieties of topological groups*, Fundamenta Mathematicae (to appear).

[15] J. von Neumann, *Almost periodic functions in a group I*, Transactions of the American Mathematical Society 36 (1934), p. 445 - 492.

Reçu par la Rédaction le 29. 2. 1972