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A TYPICAL PROPERTY OF THE SYMMETRIC DIFFERENTIAL QUOTIENT

BY

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Let R denote the set of all real numbers, I the interval $[0, 1]$, $C = C(I)$ the space of all continuous functions on I with the supremum metric, and $|M|$ the outer Lebesgue measure of a set $M \subset R$. Let

$$\bar{D}(M) = \limsup_{h \to 0} (|M \cap (-h, h)|/2h)$$

be the upper density of M at the point 0. For $f \in C$, $x \in I$, $K \in R$ put

$$S_f(x, K) = \{h; (f(x+h) - f(x-h))/2h > K\}.$$

Evans [1] constructed an $f \in C$ such that f has the approximate upper symmetric derivative $+\infty$ and the approximate lower symmetric derivative $-\infty$ at every point $x \in (0, 1)$, and he proved that the set of such functions is comeager in C. In fact, his f has

$$\bar{D}(S_f(x, K)) \geqslant 1/26 \quad \text{and} \quad \bar{D}(R \setminus S_f(x, -K)) \geqslant 1/26$$

for all x and arbitrarily large K. We strengthen his result as follows:

THEOREM. *Let F be the set of all functions $f \in C$ such that*

$$\bar{D}(S_f(x, K)) = 1 = \bar{D}(R \setminus S_f(x, -K))$$

for all $x \in (0, 1)$ and arbitrarily large K. Then F is comeager in C.

The Evans construction (op. cit.) starts with a function composed of 4 linear segments. We have to use more complex functions; we obtain them by using finite sequences of 0's and 1's with certain properties.

For easier formulation we shall treat the finite sequences of 0's and 1's as words over the alphabet $\{0, 1\}$. For a word w let $n(w)$ denote the number of symbols of w, and $w[i]$ its i-th symbol. The concatenation of words w and v is denoted by wv,

$$w^k = ww \ldots w \ (k \text{ times})$$

[1] M. J. Evans, *On continuous functions and the approximate symmetric derivative*, Colloq. Math. 31 (1974), pp. 129–136.

and w^R is composed of the symbols of w in the reversed order. Let 0 stand also for the word with a single symbol 0; similarly for 1. For h, i natural numbers, $1 \leqslant i-h+1 < i+h \leqslant n(w)$, we put

$$d_w(i, h) = \mathrm{card}(\{j \in \{1, 2, \ldots, h\};\ w[i-j+1] = 1 \text{ and } w[i+j] = 0\})/h.$$

LEMMA 1. *Let $\varepsilon > 0$. There exists a word $w = w(\varepsilon)$ such that*

$$w[1] = 1, \quad w[n(w)] = 0,$$

and for each $i \in \{1, 2, \ldots, n(w)-1\}$ there exists $h(i)$ with

$$1 \leqslant i-h(i)+1 < i+h(i) \leqslant n(w) \quad \text{and} \quad d_w(i, h(i)) < \varepsilon.$$

Proof. Let k, m be fixed natural numbers; their values will be specified later. We define a sequence of words $w_0, w_1, \ldots$ and a sequence of natural numbers $a_0, a_1, \ldots$ as follows:

$$w_0 = 1^k, \qquad a_0 = 1,$$

$$w_{j+1} = (w_j 0^{a_j})^k w_j, \quad a_{j+1} = n(w_j)+a_j.$$

We put $w = w_m 0^{n(w_m)}$.

We find the $h(i)$ for each i. Only the case $w[i] = 1$, $w[i+1] = 0$ requires discussion (otherwise $d_w(i, 1) = 0$). Denote by $z(i)$ the maximal number with the property

$$w[i+1] = w[i+2] = \ldots = w[i+z(i)] = 0.$$

We distinguish two cases:

(i) The number $z(i)$ equals a_j for some $j \in \{0, 1, \ldots, m-1\}$. One can see from the construction of the word w that each maximal segment of zeros of length a_j is preceded and followed by the word w_j. By induction we get $w_j^R = w_j$, and the recurrence relation for w_j gives

$$w_j[p+a_j] = w_j[p] \quad \text{for } p = 1, 2, \ldots, n(w_j)-a_j.$$

Combining these observations, we get

$$w[i+p] = w[i-p+1] \quad \text{for } p = a_j+1, \ldots, n(w_j).$$

Thus

$$d_w(i, n(w_j)) \leqslant a_j/n(w_j) = (n(w_{j-1})+a_{j-1})/((k+1)n(w_{j-1})+k \cdot a_{j-1}) \leqslant 1/k$$

(for $j = 0$ use directly $a_0 = 1$, $n(w_0) = k$), which can be made less than ε by choosing k suitably.

(ii) If the case (i) does not occur, then necessarily $i = n(w_m)$ and $z(i) = n(w_m)$. We put $h(i) = n(w_m)$; the required inequality for $d_w(i, h(i))$ is proved by estimating the number of ones in w. Let r_j denote the number of ones in w_j

divided by $n(w_j)$. We compute

$$r_0 = 1,$$
$$\begin{aligned} r_{j+1} &= (k+1)r_j \cdot n(w_j)/n(w_{j+1}) \\ &= r_j \cdot (k+1)n(w_j)/((k+1)n(w_j)+k \cdot a_j) \\ &= r_j \cdot \bigl(1+(k/(k+1)) \cdot a_j/n(w_j)\bigr)^{-1}, \end{aligned}$$

and since $a_j/n(w_j) \geqslant 1/(k+1)$, we have the estimation

$$d_w\bigl(i, n(w_m)\bigr) = r_m \leqslant \bigl(1+k/(k+1)^2\bigr)^{-m}.$$

Choosing m sufficiently large, we get $r_m < \varepsilon$. Thus Lemma 1 is proved.

For a function $f\colon I \to R$, a point x and $h > 0$, $0 \leqslant x-h < x+h \leqslant 1$, we put

$$d_f(x, h) = |(0, h) \cap \{t > 0;\ f(x-t) > f(x+t)\}|/h.$$

LEMMA 2. *For every $\varepsilon > 0$ there exist a continuous function $\varphi\colon I \to I$ and $h_0 > 0$ satisfying*

(i) *$\varphi = 1$ on $[0, h_0]$ and $\varphi = 0$ on $[1-h_0, 1]$;*

(ii) *for every $x \in [h_0, 1-h_0]$ there exists $h \in [h_0, 1/2]$ such that $d_\varphi(x, h) < \varepsilon$.*

Proof. Let $w = w(\varepsilon/4)$ be the word from Lemma 1 and let $N = n(w)$. Let us define a function $\psi\colon I \to \{0, 1\}$ by

$$\psi(x) = w[i] \quad \text{for } x \in \bigl[(i-1)/N, i/N\bigr),\ i = 1, 2, \ldots, N,$$

and

$$\psi(1) = 0.$$

Put $h_0 = 1/8N^2$. We show that for every $x \in [h_0, 1-h_0]$ there exists $h(x) \in [h_0, 1/2]$ with

$$d_\psi\bigl(x, h(x)\bigr) < \varepsilon/2.$$

This already proves the lemma, since if we choose φ to be a continuous function from I to I such that

$$M = \{t \in I;\ \varphi(t) \neq \psi(t)\} \subset [h_0, 1-h_0] \quad \text{and} \quad |M| < \varepsilon h_0/4,$$

then $d_\varphi\bigl(x, h(x)\bigr) < \varepsilon$ for every x and (i) is also satisfied.

Let $D = \{i/N;\ i = 1, 2, \ldots, N-1\}$. For $x \in D$ Lemma 1 guarantees the existence of $h(x) \in [1/N, 1/2]$ with $d_\psi\bigl(x, h(x)\bigr) < \varepsilon/4$. The function $\psi'(x) = \psi(x+\delta)$ and the function ψ differ on a set of measure at most $2N|\delta|$. If $x_0 \in D$ and $|x-x_0| < h_0$, then

$$\begin{aligned} &\bigl|d_\psi\bigl(x_0, h(x_0)\bigr) - d_\psi\bigl(x, h(x_0)\bigr)\bigr| \\ &\quad \leqslant \bigl|\{t \in \bigl(-h(x_0), h(x_0)\bigr);\ \psi(x_0+t) \neq \psi(x+t)\}\bigr|/h(x_0) \leqslant 2Nh_0 \cdot N \leqslant \varepsilon/4, \end{aligned}$$

so it suffices to put $h(x) = h(x_0)$. On the other hand, if a point x has distance at least h_0 from the set D, then ψ is constant in the h_0-neighborhood of x and $d_\psi(x, h_0) = 0$.

Proof of Theorem. Put

$$N_1 = \{f \in C;\ \exists x \in (0, 1), \exists K\colon \bar{D}(S_f(x, K)) < 1\};$$

we prove that N_1 is meager in C. Analogously one proves that

$$N_2 = \{f \in C;\ \exists x \in (0, 1), \exists K\colon \bar{D}(R \backslash S_f(x, -K)) < 1\}$$

is also meager, and so is $C \backslash F = N_1 \cup N_2$.

Put

$$Q_n = \{f \in C;\ \forall x \in [1/n, 1-1/n], \forall h \in (0, 1/n)\colon$$
$$|S_f(x, n) \cap (-h, h)|/2h < 1-1/n\};$$

then

$$N_1 = \bigcup_{n=2}^{\infty} Q_n$$

(indeed, if $f \in N_1$, then there exist $x \in (0, 1)$, K, $\varepsilon > 0$ and $h > 0$ such that

$$|S_f(x, K) \cap (-t, t)|/2t < 1-\varepsilon \quad \text{for every } t \in (0, h);$$

so choose

$$n > \max(K, 1/\varepsilon, 1/h, 1/x, 1/(1-x));$$

then $f \in Q_n$).

We claim that each Q_n is nowhere dense in C for $n \geqslant 2$. To this end we show that for every $f \in C$ and $r > 0$ there exist a function $g \in C$ and $\delta > 0$ so that the ball $B(g, \delta)$ is contained in $B(f, r)$ and $B(g, \delta) \cap Q_n = \emptyset$. It suffices to take the f's to be polynomials (since they are dense in C), and hence Lipschitz.

Let $f \in C$, $r < 1$ be fixed, and let L be the Lipschitz constant for f. Let φ and h_0 be as in Lemma 2 for $\varepsilon = 1/2n$. Put $K = n+L+1$ and $p = r/2K$ and define the function

$$\sigma\colon I \to [-r/2, r/2]$$

as follows:

$$\sigma(x) = r\varphi(x/p)/2 + Kx - r/2 \qquad \text{for } x \in [0, p)$$

and σ is periodic on I with a period p. Then it is easy to verify that σ is continuous and for every $x \in [p, 1-p]$ there exists $h(x) \in [p \cdot h_0, p]$ with

$$\big|(-h(x), h(x)) \cap S_\sigma(x, K)\big|/2h(x) > 1-1/2n.$$

Now put $g = f+\sigma$ and $\delta = p \cdot h_0/2n$.

Let

$$g_1 \in B(g, \delta), \qquad x \in [1/n, 1-1/n] \subset [p, 1-p].$$

If $h \in S_\sigma(x, K)$, $|h| \geqslant \delta$, then

$$\begin{aligned}(g_1(x+h) - g_1(x-h))/2h &= (f(x+h) - f(x-h))/2h \\ &\quad + (\sigma(x+h) - \sigma(x-h))/2h + \big((g_1(x+h) - g(x+h)) \\ &\quad - (g_1(x-h) - g(x-h))\big)/2h \\ &\geqslant -2Lh/2h + K - 2\delta/2h \geqslant K - L - 1 = n;\end{aligned}$$

hence $S_\sigma(x, K)\backslash(-\delta, \delta) \subset S_{g_1}(x, n)$, and thus

$$\begin{aligned}
&|(-h(x), h(x)) \cap S_{g_1}(x, n)|/2h(x) \\
&\qquad \geqslant |((-h(x), -\delta) \cup (\delta, h(x))) \cap S_\sigma(x, K)|/2h(x) \\
&\qquad \geqslant (|(-h(x), h(x)) \cap S_\sigma(x, K)| - 2\delta)/2h(x) \\
&\qquad \geqslant 1 - 1/2n - \delta/(p \cdot h_0) = 1 - 1/n.
\end{aligned}$$

We have proved that g_1 does not belong to Q_n, and so Q_n is nowhere dense.

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