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CURVATURE OF A SEMI-DIRECT EXTENSION OF A HEISENBERG TYPE NILPOTENT GROUP

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In this paper we continue the investigation of the spaces S analogous to rank one symmetric spaces defined in [1]. If N is a nilpotent group of type H (see [2]) and A a one-parameter group of dilations, then S is a solvable group $S = NA$ equipped with a suitable left-invariant metric.

However, S being a generalization of hyperbolic spaces has in general a much smaller group of isometries $I(S)$ (see [1]). This suggests that the geometry of non-classical S is probably quite different from that of a hyperbolic space. The aim of this paper is to show that the sectional curvature of S is non-positive but it is not always strictly negative in contrast with the classical cases.

We refer to [1] for definitions and most of the notation.

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1. Sectional curvature of S. Our aim is to investigate the sign of the sectional curvature $k(x, y)$, that is the sign of

$$K(x, y) = \langle \nabla_x \nabla_y y - \nabla_y \nabla_x y - \nabla_{[x,y]} y, x \rangle = k(x, y) \cdot (|x|^2 |y|^2 - \langle x, y \rangle^2)$$

for any given vectors x and y in the algebra $\mathfrak{s}$ of S. We start with a few lemmas.

LEMMA 1. *In every plane* $\sigma \subset \mathfrak{s} = V + Z + \mathfrak{a}$ *there are two independent vectors* x *and* y *of the form*

$$(1) \qquad x = v + z + re_0, \qquad y = v' + z', \qquad \langle v, v' \rangle = 0,$$

where $v, v' \in V$, $z, z' \in Z$, $Re_0 = \mathfrak{a}$, *and* V, Z, $\mathfrak{a}$ *are as in* [1]. *Moreover, if* $v \neq 0$ *and* $v' \neq 0$, *then we can assume that* $|v| = |v'| = 1$.

We omit an elementary proof.

From now on we assume that x and y are of the form (1). The following general properties of the left-invariant connection [3] are used in the proof of the next lemma. For any x, y, u belonging to the Lie algebra we have

$$\langle \nabla_x y, u\rangle = \tfrac{1}{2}(\langle [x, y], u\rangle - \langle [y, u], x\rangle + \langle [u, x], y\rangle), \tag{2}$$

$$\langle \nabla_x y, u\rangle + \langle y, \nabla_x u\rangle = 0, \tag{3}$$

$$\nabla_x y - \nabla_y x = [x, y]. \tag{4}$$

LEMMA 2. *If x and y are of the form* (1), *then*

$$K(x, y) = \langle y, [y, \nabla_x x]\rangle + |\nabla_x y|^2 - |[x, y]|^2. \tag{5}$$

Proof. By (2)–(4) we have

$$\langle \nabla_x \nabla_y y, x\rangle = \langle y, [y, \nabla_x x]\rangle, \tag{6}$$

$$-\langle \nabla_y \nabla_x y, x\rangle = |\nabla_x y|^2 - \langle \nabla_x y, [x, y]\rangle, \tag{7}$$

$$-\langle \nabla_{[x,y]} y, x\rangle = \langle [x, y], \nabla_x y\rangle - |[x, y]|^2 - \langle [[x, y], y], x\rangle. \tag{8}$$

Now, if x and y are of the form (1), then $[[x, y], y] = 0$. Thus putting (6)–(8) together we obtain (5).

LEMMA 3. *If x and y are of the form* (1), *then*

$$\begin{aligned} K(x, y) = & -\tfrac{1}{4}|v|^2|v'|^2 - \tfrac{1}{4}|z|^2|v'|^2 - \tfrac{1}{4}|z'|^2|v|^2 - |z|^2|z'|^2 + \langle z, z'\rangle^2 - \\ & -\tfrac{3}{4}|\pi\Phi(v, v')|^2 - \tfrac{1}{4}r^2|v'|^2 - r^2|z'|^2 - \tfrac{3}{2}\langle \mu(z, v), \mu(z', v')\rangle + \\ & +\tfrac{3}{2}r\langle \mu(z', v'), v\rangle, \end{aligned} \tag{9}$$

where μ and $\pi\Phi$ are as defined in [1].

Proof. By (1.1) and (4.2) of [1] we have

$$\begin{aligned} \langle y, [y, \nabla_x x]\rangle = & -\tfrac{1}{4}|v|^2|v'|^2 - \tfrac{1}{2}|z|^2|v'|^2 - \langle \mu(z', v'), \mu(z, v)\rangle - \\ & -\tfrac{1}{2}r\langle \mu(z', v'), v\rangle - \tfrac{1}{2}|v|^2|z'|^2 - |z|^2|z'|^2. \end{aligned} \tag{10}$$

Analogously,

$$\begin{aligned} |\nabla_x y|^2 = & \langle z, z'\rangle^2 + \tfrac{1}{4}|\pi\Phi(v, v')|^2 + \tfrac{1}{4}|\mu(z', v)|^2 + \tfrac{1}{4}|\mu(z, v')|^2 + \\ & +\tfrac{1}{2}\langle \mu(z', v), \mu(z, v')\rangle, \end{aligned}$$

which by (3.8) of [1] gives

$$\begin{aligned} |\nabla_x y|^2 = & \langle z, z'\rangle^2 + \tfrac{1}{4}|\pi\Phi(v, v')|^2 + \tfrac{1}{4}|z'|^2|v|^2 + \tfrac{1}{4}|z|^2|v'|^2 - \\ & -\tfrac{1}{2}\langle \mu(z, v), \mu(z', v')\rangle \end{aligned} \tag{11}$$

and, finally,

$$(12)\qquad -|[x, y]|^2 = \tfrac{1}{4}r^2|v'|^2 - |\pi\Phi(v, v')|^2 - r^2|z'|^2 + 2r\langle\mu(z', v'), v\rangle.$$

Now (10)–(12) imply (9).

THEOREM. *The sectional curvature of S is non-positive.*

Proof. Since it is easy to see that $K(x, y) < 0$ when $v = 0$ or $v' = 0$, we assume $|v| = |v'| = 1$. Let $z' = z_1 + z''$, where $z_1 = \alpha z$ for an $\alpha \in R$ and $\langle z'', z\rangle = 0$. Then

$$K(x, y) = W_1(x, y) + W_2(x, y),$$

where

$$W_1(x, y) = -\tfrac{1}{4} - \tfrac{1}{4}|z|^2 - \tfrac{1}{4}|z''|^2 - |z|^2|z''|^2 - \tfrac{1}{4}r^2 - r^2|z''|^2 - \\ -\tfrac{3}{2}\langle\mu(z, v), \mu(z'', v')\rangle + \tfrac{3}{2}r\langle\mu(z'', v'), v\rangle,$$

$$W_2(x, y) = -\tfrac{1}{4}|z_1|^2 - r^2|z_1|^2 - \tfrac{3}{4}|\pi\Phi(v, v')|^2 - \tfrac{3}{2}\langle\mu(z, v), \mu(z_1, v')\rangle + \\ +\tfrac{3}{2}r\langle\mu(z_1, v'), v\rangle.$$

First we shall show that $W_1 \leqslant 0$. By the Schwarz inequality, our assumptions on norms and $\langle v, \mu(z, v)\rangle = 0$ we obtain

$$(13)\qquad |\tfrac{3}{2}\langle\mu(z'', v'), rv - \mu(z, v)\rangle| \leqslant \tfrac{3}{2}|z''|(r^2 + |z|^2)^{1/2}.$$

Moreover,

$$(14)\quad -\tfrac{1}{4}(r^2 + |z|^2) - \tfrac{1}{4}|z''|^2 + \tfrac{1}{2}|z''|(r^2 + |z|^2)^{1/2} = -\tfrac{1}{4}((r^2 + |z|^2)^{1/2} - |z''|)^2 \leqslant 0.$$

For every non-negative real a we have $\sqrt{a} \leqslant a + \frac{1}{4}$. Hence

$$(15)\qquad -\tfrac{1}{4} - |z|^2|z''|^2 - r^2|z''|^2 + |z''|(r^2 + |z|^2)^{1/2} \leqslant 0.$$

From (13)–(15) we infer now that $W_1 \leqslant 0$.

Notice that

$$(16)\qquad -\tfrac{3}{4}|\pi\Phi(v', v)|^2 = \tfrac{3}{4}|rz_1|^2 - \tfrac{3}{2}r\langle z_1, \pi\Phi(v', v)\rangle - \tfrac{3}{4}|rz_1 - \pi\Phi(v', v)|^2$$

and by (3.1) of [1] we get

$$(17)\qquad \langle\mu(z, v), \mu(z_1, v')\rangle = 0.$$

Putting (16) and (17) into W_2 we obtain

$$W_2(x, y) = -\tfrac{1}{4}|z_1|^2 - \tfrac{1}{4}r^2|z_1|^2 - \tfrac{3}{4}|rz_1 - \pi\Phi(v', v)|^2 \leqslant 0,$$

which completes the proof.

The proof of the Theorem shows that the sectional curvature of the plane σ vanishes if and only if there is a basis $x = v + z + re_0$, $y = v' + z'$ of σ with the following properties:

$$(18a)\qquad \langle v, v'\rangle = 0,$$

$$(18b)\qquad |v| = |v'| = 1,$$

$$(18c)\qquad \pi\Phi(v', v) = 0,$$

$$(18d)\qquad rv - \mu(z, v) = \alpha\mu(z', v') \quad \text{for an } \alpha > 0,$$

$$(18e)\qquad \langle z, z'\rangle = 0,$$

$$(18f)\qquad r^2 + |z|^2 = |z'|^2,$$

$$(18g)\qquad |z'|^2(|z|^2 + r^2) = \tfrac{1}{4}.$$

We shall reformulate formulas (18) to obtain a shorter list of equivalent conditions. By (18c) and (18d) we have

$$r|v|^2 = \langle rv - \mu(z, v), v\rangle = \langle \alpha\mu(z', v'), v\rangle = 0.$$

Hence $r = 0$. Comparing the norms of $\mu(z, v)$ and $\mu(z', v')$ and taking into account (18b) and (18f) we see that $\alpha = 1$. Moreover, (18e) follows from (18a), (18d), and (3.2) of [1]:

$$\langle z, z'\rangle = \langle \mu(z, v), \mu(z', v)\rangle = -\langle \mu(z', v'), \mu(z', v)\rangle = 0.$$

Thus conditions (18) can be written in the equivalent forms:

$$(19)\qquad \begin{gathered} \langle v, v'\rangle = 0, \quad |v| = |v'| = 1, \quad \pi\Phi(v', v) = 0, \\ -\mu(z, v) = \mu(z', v'), \quad |z|^2 = |z'|^2 = \tfrac{1}{2}. \end{gathered}$$

Let $v \in V$ and $D(v) = \{\mu(z, v)\colon z \in Z\}$. Our considerations above lead to the following

PROPOSITION. *There is a plane in* $\mathfrak{s}$ *with sectional curvature* 0 *if and only if there are* $v, v' \in V$ *such that*

$$(20)\qquad \langle v, v'\rangle = 0, \quad \pi\Phi(v', v) = 0, \quad D(v) \cap D(v') \neq \{0\}.$$

We omit an obvious proof.

EXAMPLE. The sectional curvature of a rank one symmetric space is strictly negative. Thus (20) cannot be satisfied, as we can also see this directly by Proposition 4.1 of [1]. But already in the first non-classical S of the lowest possible dimension such v and v' do exist. To verify this we denote by H the field of quaternions. Let $V = H$ and $U = \mathrm{lin}_{\mathbf{R}}(1, i, j)$. We equip V and U with the standard inner product

$$\langle q, q'\rangle = 4\,\mathrm{Re}(\bar{q}q'), \qquad q, q' \in H.$$

Then

$$\mu\colon \mathrm{lin}_{\mathbf{R}}(1, i, j) \times H \to H$$

defined by

$$\mu(p, q) = 2qp$$

is a composition of quadratic forms (cf. [2]). Denote by $\varphi\colon H \to \mathrm{lin}_{\boldsymbol{R}}(i, j)$ the orthogonal projection. Then $\mathfrak{n} = H \times \mathrm{lin}_{\boldsymbol{R}}(i, j)$ with the bracket $[q, q'] = 2\varphi(\bar{q}q')$ is a nilpotent Lie algebra of type H. Now $D(1) = D(k) = \mathrm{lin}_{\boldsymbol{R}}(i, j)$. Hence $v = 1$ and $v' = k$ satisfy (20). Therefore, if S is the extension of the nilpotent group $H \times \mathrm{lin}_{\boldsymbol{R}}(i, j)$, then its sectional curvature cannot be strictly negative.

Similarly taking the octonions in place of the quaternions we obtain other examples of spaces S with not strictly negative curvature.

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