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REMARKS ON INVARIANT MEASURES IN METRIC SPACES

BY

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1. Introduction and results. Let X be a metric space and let $\mathscr{B}$ be the class of Borel subsets of X, i.e., the σ-algebra generated by the open sets. A *measure* over X is a countably additive function $\mu\colon \mathscr{B} \to [0, \infty]$. A measure μ is *regular* if

$$\mu(A) = \inf\{\mu(U)\colon A \subseteq U \text{ and } U \text{ is open in } X\} \quad \text{for all } A \in \mathscr{B}$$

(for a discussion of this concept in a context similar to ours see [2]).

For many compact spaces X there is no measure μ over X such that $\mu(X) = 1$ and $\mu(A) = \mu(B)$ whenever A is isometric to B, e.g., if X is infinite countable. But our Theorem 1 shows that this is possible if we restrict this condition to A and B open in X.

We define an *entropy* E for an arbitrary metric space X by

$$E(C, t) = \min\{\operatorname{card}\mathscr{K}\colon \mathscr{K} \text{ is a covering of } C \text{ with sets of diameters } < t\},$$

where C runs over all compact subsets of X and $0 < t \leqslant 1$.

A compact set $\emptyset \neq C \subseteq X$ will be called *thick* in X if there exists an open set $U \subseteq X$ and a finite constant a such that $C \subseteq U$ and $E(D, t) \leqslant aE(C, t)$ for all compact $D \subseteq U$ and all $t \in (0, 1]$.

Examples. 1. If C is compact and open in X, then C is thick in X.

2. If G is a topological group with a left (right) invariant metric and $C \subseteq G$ is a compact set with non-empty interior, then C is thick in G. This follows since some open set including C can be covered with finitely many left (right) translates of C.

THEOREM 1. *If X is a metric space and C is a compact set thick in X, then there exists a regular measure μ over X such that $\mu(C) = 1$ and $\mu(A) = \mu(B)$ whenever A and B are isometric open sets.*

It follows that this measure μ has additional properties: If A and B are isometric Borel sets for which there exists an isometry which can be extended to some open sets including A and B, then $\mu(A) = \mu(B)$. In particular, μ is invariant under all isometries of X onto itself.

In our proof of Theorem 1 we use the axiom of choice for uncountable families of sets and this proof is similar to that of Banach of the existence of Haar measure [1]. We have not fully explored the possibility that the proof of Loomis [8], where the existence and unicity of measures is proved (under stronger assumptions) without using this axiom, can be adapted to our context. However, in our case the unicity may fail. For example, if X is a union of two circular curves with different radii in the plane and $C = X$, then there exist many measures satisfying the conditions of Theorem 1. The special measure μ which will be constructed in our proof of Theorem 1 apparently depends on a parameter $\mathcal{F}$ but we do not know if it really does (**P 919**).

By Example 2 and the theorem of Kakutani [6], Theorem 1 yields the existence of Haar measures for locally compact first countable groups.

For more information and references related to Theorem 1 see [3], [5], [7], [8] and [12].

Let us return to the question of the existence of a measure μ over X with $\mu(C) = 1$ and such that $\mu(A) = \mu(B)$ for every $A, B \in \mathcal{B}$, where A is isometric to B. The following condition on a compact set $C \subseteq X$ secures the existence of such a measure:

(Γ) There exists an $a > 0$ such that, for every finite sequence of numbers $t_1, \ldots, t_m$ in $(0, 1]$ for which there exists a covering $A_1, \ldots, A_m$ of C with $\operatorname{diam}(A_i) < t_i$ for $i = 1, \ldots, m$, we have

$$\sum_{i=1}^{m} \frac{1}{E(C, t_i)} \geqslant a.$$

Indeed, under this condition, putting $h(t) = 1/E(C, t)$, the Hausdorff h-measure μ_h over X (see [4], p. 30-31) satisfies $a \leqslant \mu_h(C) \leqslant 1$. Hence $\mu(Y) = \mu_h(Y)/\mu_h(C)$ is an invariant measure, as required.

Unlike thickness condition (Γ) does not depend on the position of C in X. Unfortunately, it is not clear which compact spaces C satisfy (Γ) (see [13]). n-dimensional cubes or parallelepipeds satisfy (Γ) but many compact sets in $\boldsymbol{R}^n$ do not (e.g., infinite countable ones).

P 920. Does a compact parallelepiped in the Hilbert space l^2, say $\prod_{n=1}^{\infty} [0, 1/n]$, satisfy ($\Gamma$)?

The best we can get from Theorem 1 is that, for a large class of natural metrisations of the Hilbert cube $[0, 1]^\omega$, the standard product measure is equal on isometric open sets.

P 921. Does a set C as in Example 2 satisfy (Γ)?

An affirmative answer to the last question would imply that if ϱ is a left (right) invariant metric in a locally compact group G, then isometric Borel subsets of G have equal Haar measure. But we can prove only the following much weaker Corollary to Theorem 1:

COROLLARY. *If ϱ is a metric in $\boldsymbol{R}^n$ consistent with the usual topology and invariant under translations, $A, B \subseteq \boldsymbol{R}^n$ are open sets and A is isometric to B with respect to ϱ, then A and B have the same Lebesgue measure.*

Since the unit cube I^n is thick in $(\boldsymbol{R}^n, \varrho)$ (by Example 2), this Corollary follows from Theorem 1 and from the following known proposition:

PROPOSITION. *If μ is a measure over $\boldsymbol{R}^n$ such that $\mu(I^n) = 1$ and $\mu(V+t) = \mu(V)$ for every open set V and every vector $t \in \boldsymbol{R}^n$, then $\mu(A)$ is the Lebesgue measure of A for every $A \in \mathscr{B}$.*

P 922. Does I^n in $(\boldsymbol{R}^n, \varrho)$ satisfy (Γ)? (If so, then the conclusion of the Corollary could be extended to all $A, B \in \mathscr{B}$.) For a result related to the Corollary see [15].

Our second theorem concerns paradoxical decompositions. A metric space X is said to *have a paradoxical decomposition* with parts of class $\mathscr{M}$ if there exist three finite partitions of X into disjoint sets of class $\mathscr{M}$,

$$X = A_1 \cup \ldots \cup A_r, \quad X = B_1 \cup \ldots \cup B_s$$

and

$$X = A_1' \cup \ldots \cup A_r' \cup B_1' \cup \ldots \cup B_s',$$

where A_i is isometric to A_i' and B_j is isometric to B_j' for $i = 1, \ldots, r$ and $j = 1, \ldots, s$.

It is known (see, e.g., [11]) that the real line $\boldsymbol{R}$ and the plane $\boldsymbol{R}^2$ have no paradoxical decompositions, while $\boldsymbol{R}^n$ for $n \geqslant 3$ has such decompositions. But we will prove here the following

THEOREM 2. *The space $\boldsymbol{R}^n$ has no paradoxical decompositions with Lebesgue measurable parts.*

By a theorem of Tarski (see [16] or [11]) and the elementary fact that every isometry of a subset of $\boldsymbol{R}^n$ onto another can be extended to as isometry of $\boldsymbol{R}^n$ onto itself, Theorem 2 is equivalent to the following

THEOREM 3. *There exists a finitely additive measure ν on the field of Lebesgue measurable subsets of $\boldsymbol{R}^n$ which is invariant under isometries and satisfies $\nu(\boldsymbol{R}^n) = 1$.*

It follows that $\nu(A) = 0$ for every bounded set A.

We do not know any proof of Theorem 3 in which there would not be used the axiom of choice for uncountable families of sets. We shall prove directly the seemingly stronger Theorem 3. But one can produce a proof of Theorem 2 in which one does not use the axiom of choice by analyzing our proof along the lines indicated by Morse [10] (this is based on the observation that the Hahn-Banach theorem for separable Banach spaces does not require the axiom of choice). A more direct proof of this sort was communicated to us by Roy O. Davies and is included below with his permission.

For more information and references related to Theorems 2 and 3 and fascinating open problems of Marczewski and Ruziewicz see [11].

A similar open problem of S. M. Ulam (*Measure and set theory*, a film in the Mathematical Association of America Individual Lecture Series, or *The Scottish Book*, Problem 2) is the following:

Is there a finitely additive measure μ on the class of Borel sets in a compact metric space X with $\mu(X) = 1$, which is equal on isometric sets? An equivalent question (by the theorem of Tarski l.c.) is whether no such X has paradoxical decompositions with Borel parts. Roy O. Davies and O. Ostaszewski have proved that for countable X the answer is positive.

2. Preliminaries on generalized limits. Let $[0, \infty]$ be endowed with its natural compact topology. Let ω be the set of non-negative integers and let $\mathscr{F}$ be an ultrafilter of subsets of ω. For every sequence $x_n \in [0, \infty]$, $n < \omega$, we define $\lim\limits_{n\to\mathscr{F}} x_n$ to be the unique $x \in [0, \infty]$ such that every neighbourhood V of x has the property $\{n\colon x_n \in V\} \in \mathscr{F}$. This limit (for other related limits see [9]) has the property that if $m < \omega$ and $f\colon [0, \infty]^m \to [0, \infty]$ is a continuous function and $x_n^{(i)} \in [0, \infty]$ for all $n < \omega$ and $i = 1, \ldots, m$, then

$$\lim_{n\to\mathscr{F}} f(x_n^{(1)}, \ldots, x_n^{(m)}) = f(\lim_{n\to\mathscr{F}} x_n^{(1)}, \ldots, \lim_{n\to\mathscr{F}} x_n^{(m)}).$$

In particular,

$$\lim_{n\to\mathscr{F}} (x_n + y_n) = \lim_{n\to\mathscr{F}} x_n + \lim_{n\to\mathscr{F}} y_n,$$

where $+$ is extended in the natural way over $[0, \infty]$, i.e., $x + \infty = \infty$.

Finally, if $\mathscr{F}$ is a non-principal ultrafilter, then this limit is a conservative extension of the ordinary limit.

3. Proof of Theorem 1. Let $\mathscr{C}$ be the class of compact subsets of X. Thus $C \in \mathscr{C}$.

LEMMA A. *There exists a function $\lambda\colon \mathscr{C} \to [0, \infty]$ satisfying, for all $A, B \in \mathscr{C}$, the following conditions:*

(o) $\lambda(C) = 1$;

(i) *there exists an open set $U \subseteq X$ and a finite constant a such that $C \subseteq U$ and $\lambda(A) < a$ for all $A \subseteq U$;*

(ii) $\lambda(A \cup B) \leqslant \lambda(A) + \lambda(B)$;

(iii) $\lambda(A \cup B) = \lambda(A) + \lambda(B)$ *if* $A \cap B = \emptyset$;

(iv) $\lambda(A) = \lambda(B)$ *if A is isometric to B.*

Proof. We put, for all $A \in \mathscr{C}$,

$$\lambda(A) = \lim_{n \to \mathscr{F}} \frac{E(A, 1/n)}{E(C, 1/n)},$$

where $\mathscr{F}$ is a non-principal ultrafilter of sets in ω and E is the entropy defined in Section 1. All properties (o)-(iv) are visible from this definition and the assumption that C is thick.

For every open set $U \subseteq X$, we put

$$\mu_0(U) = \sup\{\lambda(A) : A \in \mathscr{C},\ A \subseteq U\},$$

and, for every set $Y \subseteq X$, we put

$$\mu^*(Y) = \inf\{\mu_0(U) : Y \subseteq U \text{ and } U \text{ is open in } X\}.$$

Lemma B. *μ^* satisfies the following conditions:*

(o) $\mu^*(\emptyset) = 0$;

(i) $1 \leqslant \mu^*(C) < \infty$;

(ii) $\mu^*(Y) \leqslant \mu^*(Z)$ *if* $Y \subseteq Z$;

(iii) $\mu^*\big(\bigcup_{n<\omega} Y_n\big) \leqslant \sum_{n<\omega} \mu^*(Y_n)$;

(iv) $\mu^*(Y \cup Z) = \mu^*(Y) + \mu^*(Z)$ *if*

$$(*) \qquad \inf\{\operatorname{dist}(y, z) : y \in Y,\ z \in Z\} > 0;$$

(v) $\mu^*(U) = \mu^*(V)$ *if U and V are open and U is isometric to V.*

Proof. By Lemma A (o) and (iii), we have $\lambda(\emptyset) = 0$ and (o) follows.

(i) is obvious from Lemma A (o) and (i).

(ii) is obvious.

(iii) First, we show that

$$(**) \qquad \mu_0\big(\bigcup_{n<\omega} U_n\big) \leqslant \sum_{n<\omega} \mu_0(U_n)$$

if U_n are open. Let

$$A \subseteq \bigcup_{n<\omega} U_n \quad \text{and} \quad A \in \mathscr{C}.$$

Then there exists an integer m such that

$$A \subseteq \bigcup_{n<m} U_n$$

and there are sets $A_0, \ldots, A_{m-1}$ in $\mathscr{C}$ such that

$$A = \bigcup_{n<m} A_n \quad \text{and} \quad A_n \subseteq U_n \text{ for } n < m.$$

Thus $(**)$ follows from Lemma A (ii).

Now, to prove (iii) we choose open sets $U_n \supseteq Y_n$ such that

$$\mu_0(U_n) \leqslant \mu^*(Y_n) + \varepsilon/2^{n+1}.$$

Then, by (**),

$$\mu^*\Big(\bigcup_{n<\omega} Y_n\Big) \leqslant \mu_0\Big(\bigcup_{n<\omega} U_n\Big) \leqslant \sum_{n<\omega} \mu_0(U_n) \leqslant \sum_{n<\omega} \mu^*(Y_n) + \varepsilon,$$

and (iii) follows.

(iv) By (*) it follows that there are open sets $U \supseteq Y$ and $V \supseteq Z$ with $U \cap V = \emptyset$. For every $A \subseteq U \cup V$ and $A \in \mathscr{C}$, we have $A \cap U \in \mathscr{C}$ and $A \cap V \in \mathscr{C}$. Hence Lemma A (iii) yields $\mu^*(Y \cup Z) \geqslant \mu^*(Y) + \mu^*(Z)$ and, by B (iii), we get (iv).

(v) is obvious by Lemma A (iv), which concludes the proof.

Proof of Theorem 1. By Lemma B (o), (ii), (iii) and (iv), μ^* is a Carathéodory outer measure on X. Hence (see [14], Proposition 32, p. 285) all Borel sets are measurable with respect to μ^*. By Lemma B (i), we can write

$$\mu(A) = \mu^*(A)/\mu^*(C) \quad \text{for all } A \in \mathscr{B}.$$

Hence $\mu(C) = 1$ and μ is a measure over X (by Theorem 1, p. 251, in [14]). By the definition of μ^*, the measure μ is regular and, by Lemma B (v), the last requirement of Theorem 1 is satisfied.

4. Proof of Theorem 3. Let $\mathscr{L}$ be the class of Lebesgue measurable sets in $\boldsymbol{R}^n$ and, for every $A \in \mathscr{L}$, let $|A|$ be the n-dimensional Lebesgue measure of A. For every $r < \omega$, we let S_r be a ball of radius r around the origin in $\boldsymbol{R}^n$. For every $A \in \mathscr{L}$, we put

$$\nu(A) = \lim_{r \to \mathscr{F}} \frac{|A \cap S_r|}{|S_r|},$$

where $\mathscr{F}$ is a non-principal ultrafilter of subsets of ω. It is obvious that ν is a finitely additive measure over $\mathscr{L}$ which is invariant under such isometries of $\boldsymbol{R}^n$ which do not move the origin. It remains to prove that γ is invariant under translations. But this obviously follows from the fact that, for every vector v,

$$\lim_{r \to \infty} \frac{|S_r \Delta (S_r + v)|}{|S_r|} = 0,$$

where Δ is the symmetric difference of sets and $S_r + v = \{x + v \colon x \in S_r\}$.

5. A direct proof of Theorem 2 without axiom of choice (Roy O. Davies). Suppose there is a paradoxical decomposition of $\boldsymbol{R}^n$ with

Lebesgue measurable parts

$$\mathbf{R}^n = A_1 \cup \ldots \cup A_r, \qquad \mathbf{R}^n = B_1 \cup \ldots \cup B_s,$$

$$\mathbf{R}^n = A_1' \cup \ldots \cup A_r' \cup B_1' \cup \ldots \cup B_s',$$

and let the corresponding isometries be

$$\Phi_\varrho\colon A_\varrho \to A_\varrho' \quad \text{and} \quad \Psi_\sigma\colon B_\sigma \to B_\sigma'.$$

Choose R so large that

$$|S_R \Delta \Phi_\varrho S_R| < |S_R|/2r \qquad (\varrho = 1, \ldots, r),$$

$$|S_R \Delta \Psi_\sigma S_R| < |S_R|/2s \qquad (\sigma = 1, \ldots, s).$$

We get an immediate contradiction by looking at the three decompositions of S_R.

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