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# A CHARACTERIZATION OF SEMIGROUPS WHICH ARE SEMILATTICES OF GROUPS

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Let $S$ be a semigroup(1). Following the terminology of Clifford [1], [2] we shall say that $S$ is a *semilattice of groups* if $S$ is a class sum of a set $\{G_a, a \in I\}$ of mutually disjoint groups $G_a$ such that, for every $a$ and $\beta$ in $I$, the products $G_a G_\beta$ and $G_\beta G_a$ are both contained in the same $G_\gamma$ $(\gamma \in I)$.

The following characterization of semigroups which are semilattices of groups is well-known (see [1] and [3]):

THEOREM 1. *A semigroup $S$ is a semilattice of groups if and only if it satisfies the following two conditions*:

(1) $a \in Sa^2 \cap a^2 S$ *for every* $a \in S$;

(2) *if $e$ and $f$ are idempotent elements of $S$, then* $ef = fe$.

It is also well-known (see [2]) that the condition that a semigroup $S$ be a semilattice of groups is equivalent to the conjunction of any two of the following conditions:

(I) $S$ is a union of groups.

(II) $S$ is an inverse semigroup.

(III) Every one-sided ideal of $S$ is a two-sided ideal.

We shall also use the following result (see [2]):

THEOREM 2. *A semigroup $S$ is regular if and only if* $R \cap L = RL$ *for every left ideal $L$ and every right ideal $R$ of $S$.*

In this note we give another characterization of semigroups which are semilattices of groups. Our characterization reads as follows:

THEOREM 3. *A semigroup $S$ is a semilattice of groups if and only if it satisfies the following two conditions*:

(i) *$S$ is regular*;

(ii) $RL = LR$ *for every left ideal $L$ and for every right ideal $R$ of $S$.*

(1) We adopt the terminology of Clifford and Preston [2].

Proof. Necessity. Let $S$ be a semigroup which is a semilattice of groups. Then, by conditions (II) and (III), $S$ is an inverse semigroup and every one-sided ideal of $S$ is a two-sided ideal. This and Theorem 2 imply that $S$ is regular and the relation $AB = BA$ holds for every two ideals $A, B$ of $S$.

Sufficiency. Let $S$ be a semigroup which satisfies conditions (i) and (ii). Then, by Theorem 2, for any element $a$ in $S$ we have

$$(a)_L = Sa = SaS$$

and

$$(a)_R = aS = SaS.$$

Therefore $aS = Sa$ for each element $a$ of $S$, i.e. $S$ is a centric semigroup. Since $a \in aSa$ by (i), it follows that

$$a \in Sa^2 \cap a^2S,$$

that is condition (1) of Theorem 1 is valid. On the other hand, the idempotent elements of a centric semigroup lie in the center (see [2]), and thus it follows that condition (2) of Theorem 1 is also valid.

We have some easy consequences of Theorem 3.

COROLLARY 1. *Any commutative regular semigroup is a semilattice of groups.*

COROLLARY 2. *Let $S$ be a semigroup which is a semilattice of groups. Then, for every ideal $I$ of $S$,*

$$IS = I = SI,$$

*that is, the semigroup $S$ reproduces its ideals (in the sense of G. Szász [5]).*

COROLLARY 3. *Let $S$ be a semigroup which is a semilattice of groups. Then the set of all ideals of $S$ is a semilattice under the multiplication of subsets.*

For other characterizations of semigroups which are semilattices of groups see the author's paper [4].

## *REFERENCES*

[1] A. H. Clifford, *Bands of semigroups*, Proceedings of the American Mathematical Society 5 (1954), p. 499-504.

[2] — and G. B. Preston, *The algebraic theory of semigroups I-II*, American Mathematical Society, Providence, R. I., 1961, 1967.

[3] R. Croisot, *Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples*, Annales Scientifiques de l'École Normale Supérieure (3) 70 (1953), p. 361-379.

[4] S. Lajos, *On semigroups which are semilattices of groups*, Acta Scientiarum Mathematicarum (to appear).
[5] G. Szász, *Über Halbgruppen, die ihre Ideale reproduzieren*, ibidem 27 (1966), p. 141-145.

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