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A MINIMALIZATION OF 0-DIMENSIONAL METRIC SPACES

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A completely regular space with the topology τ is said to be *minimal completely regular* if, for each completely regular topology τ' on X, $\tau' \subset \tau$ implies $\tau' = \tau$. Viglino [3] has proved that each minimal completely regular space is compact Hausdorff. In view of that, the following question arises: what completely regular spaces can be minimalized to compact Hausdorff ones. Since for such spaces the Baire Category Theorem must be valid, the space of rationals, for example, cannot be minimalized. However, the space of irrationals can be minimalized, since it is homeomorphic to the infinite product of natural numbers and it is easy to observe that a locally compact space can be minimalized and that the product of minimalizable spaces is minimalizable.

Each completely metrizable space satisfies the Baire Category Theorem. Hence one might suspect that such a space can be minimalized. The answer is, however, negative: Aarts et al. [1] proved the existence of a 1-dimensional, separable and complete metric space X such that there is no continuous 1-1 map from X onto any compact Hausdorff space Y.

Now, it seems natural to ask whether each 0-dimensional (in the sense of dim) complete metric space can be minimalized. The answer is positive.

THEOREM. *Each* 0-*dimensional complete metric space X can be minimalized to a compact Hausdorff one. If, in addition, X is separable, then it can be minimalized to a compact metric space.*

Proof. If a space X is 0-dimensional and complete metric, then X has a complete uniformity with a countable base consisting of coverings of order one. Hence (see [2]) X is homeomorphic to the limit of a countable inverse system $\{X_m, \pi_n^m, N\}$ of discrete spaces X_m and, if X is separable, then $\operatorname{card} X_m \leqslant \aleph_0$, $m \geqslant 0$.

We shall change in each space X_n the (discrete) topology in a way such that the new spaces X_n^* will be compact Hausdorff and the bonding maps $\pi_n^m\colon X_m^* \to X_n^*$ will be continuous.

To do this we first classify points of each X_n, $n = 0, 1, \ldots$

(0) Without loss of generality we may assume that $X_0 = \{p_0\}$, and call p_0 to be a 0-point.

(1) Choose a point $p_1 \in X_1 = (\pi_0^1)^{-1}(p_0)$ and call it a 1-point of X_1, and all other points $x \in X_1$, $x \neq p_1$, call 0-points of X_1.

(n) Suppose that points of each X_k are classified for $k < n$. For each $x \in X_{n-1}$ choose a point $p_n^x \in (\pi_{n-1}^n)^{-1}(x)$. Call a point $y \in X_n$ to be a 0-point if $y \neq p_n^x$ for each $x \in X_{n-1}$, and an m-point of X_n, $m \leqslant n$, if $y = p_n^x$ and x is an $(m-1)$-point for some $x \in X_{n-1}$.

Now we may define a new topology on X_n. Let $\bar{y}_n \in X_n$. Define a basic neighbourhood of $\bar{y}_n$ to be a set of the form

(0) $\{\bar{y}_n\}$ if y_n is a 0-point of X_n

or of the form

(k) $(\pi_{n-1}^n)^{-1}\Big[\ldots\Big[(\pi_{n+1-k}^{n+2-k})^{-1}\,[(\pi_{n-k}^{n+1-k})^{-1}(\bar{y}_{n-k})]\setminus A_{n+1-k}\Big]\ldots\Big]\setminus A_n$ if $\bar{y}_n$ is a k-point of X_n, where A_{n-j} are arbitrary finite subsets of X_{n-j} such that

$$\bar{y}_{n+i+1-k} \notin A_{n+i+1-k} \subset (\pi_{n+i-k}^{n+i+1-k})^{-1}(\bar{y}_{n+i-k}),$$

where $\bar{y}_{n+i-k}$ is an i-point of X_{n+i-k} such that $\pi_{n+i-k}^n(\bar{y}_n) = \bar{y}_{n+i-k}$, $k \geqslant i \geqslant 0$.

Denote by X_n the space obtained from X_n by inducing the new topology. Spaces X_n^* are compact Hausdorff, and if $\operatorname{card} X_n \leqslant \aleph_0$, then X_n^* is compact metric. Maps $\pi_n^m\colon X_m^* \to X_n^*$ are continuous for each $m \geqslant n \geqslant 0$. The limit X^* of the inverse system $\{X_n, \pi_n^m, N\}$ is compact Hausdorff (compact metric), hence X^* is a compact Hausdorff (compact metric) minimalization of X.

REFERENCES

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